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A *m*-HESSIAN NORM FOR A CLASS OF DELTA-SUBHARMONIC FUNCTIONS

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Abstract: *We consider differences of m-subharmonic functions in the energy class* $\mathcal{F}_m$ *as a linear space, and equip this space with a norm, depending on the generalized complex m-Hessian operator, turning the linear space into a Banach space* $\delta\mathcal{F}_m$*. Fundamental topological questions for this space is studied, and we prove that* $\delta\mathcal{F}_m$ *is not separable.*

Keywords: *Convergence in capacity; plurisubharmonic functions; m-subharmonic functions; complex m-Hessian operator; complex Monge–Ampère operator; m-hyperconvex domain; Radon measure; positive measures.*

1. Introduction and notations

Recently, to extend the domain of definition of this operator for plurisubharmonic functions which are not neccessary to be locally bounded, in [5], [6] Cegrell introduced and investigated the classes $\mathcal{E}_0(\Omega), \mathcal{F}_p(\Omega), \mathcal{E}_p(\Omega), \mathcal{F}(\Omega), \mathcal{E}(\Omega)$ on which the complex Monge-Ampère operator is well defined. He has developed pluripotential theory on these classes. To extend the class of plurisubharmonic functions and to study a class of the complex differential operators more general than the Monge-Ampère operator, in [2] and [9], the authors introduced *m*-subharmonic functions and studied the complex Hessian operator. They also were interested in the complex Hessian equations in $\mathbb{C}^n$ and on compact Kahler manifolds. Continuing to study the complex Hessian operator for *m*-subharmonic functions which may be not locally bounded, in recent preprint, Lu Hoang Chinh introduced the Cegrell classes $\mathcal{E}_m^0(\Omega), \mathcal{F}_m(\Omega)$ and $\mathcal{E}_m(\Omega)$ associated to *m*-subharmonic functions (for details, see [8]) and has proved the complex Hessian operator is well defined on these classes.

On the other hand, convex, subharmonic and plurisubharmonic functions are all convex cones in some larger linear space. Given any such cone, K say, we can investigate the space of differences from this cone δK. Such studies are often motivated by algebraic completion of the cone, and differences of convex functions were considered by F. Riesz in as early as 1911. The δ - convex functions, or d.c. functions as they sometimes are denoted, were studied by Kiselman [12], and Cegrell [4], and have been given attention in many areas ranging from nonsmooth optimization to super-reflexive Banach spaces [7]. The δ -subharmonic where first given a

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systematic treatise in [1]. The class δ -plurisubharmonic functions were studied by Cegrell [3], and Kiselman [12], where the topology was defined by neighbourhood basis of the form $(U \cap \mathcal{PSH}) - (U \cap \mathcal{PSH})$, U a neighbourhood of the origin L^1_{loc}.

In [13], we gave a subset of *m*-subharmonic functions, the subclasses $\delta\mathcal{E}^0_m(\Omega)$ and $\delta(SH_m \cap L^\infty)(\Omega)$ of $\delta - SH_m(\Omega)$. In this paper we will give and study a subset of delta *m*-subharmonic functions, $\delta\mathcal{F}_m(\Omega)$. Let K be a hyperconvex domain in $\mathbb{C}^n$, then $\delta\mathcal{F}_m(\Omega)$ is a convex cones in the linear space $\delta\mathcal{F}_m(\Omega)$. Where $\delta\mathcal{F}_m(\Omega)$ denote the set of functions $u \in \delta\mathcal{F}_m(\Omega)$ that can be written as $u = u_1 - u_2$, where $u_i \in \delta\mathcal{F}_m(\Omega)$. We will define a norm, depending on the *m*-Hessian operator, for functions in this class and discuss some of the topological questions that this norm raises.

For convenience we will denote the class of negative *m*-subharmonic functions on a domain Ω by $\mathcal{SH}^-_m(\Omega)$, and as in [8] we will denote the class of bounded *m*-subharmonic functions with boundary value zero and finite total *m*-Hessian mass by $\mathcal{E}^0_m(\Omega)$.

For the notation of the so called *energy class* $\mathcal{F}_m(\Omega)$ on a hyperconvex domain Ω we refer to the paper [8]. As for now we remind the reader that the generalized complex *m*-Hessian operator is well defined in $\mathcal{F}_m(\Omega)$ and functions from $\mathcal{F}_m(\Omega)$ has finite total *m*-Hessian mass.

2. Denefinition of the norm

Denefiniton 2.1. *Let Ω be a hyperconvex set in $\mathbb{C}^n$. Assume that $u \in \delta\mathcal{F}_m(\Omega)$ then we define the norm of u to be:*

$$\|u\|_m = \inf_{\substack{u_1 - u_2 = u \\ u_1, u_2 \in \mathcal{F}_m(\Omega)}} \left[\left(\int_\Omega H_m(u_1 + u_2) \right)^{\frac{1}{m}} \right].$$

The following Lemma will be used repeatedly (see [10]).

Lemma 2.2. *Suppose* $u_1, u_2 \in \mathcal{E}^0_m(\Omega), 1 \le p, q < m$ *and* $T = -hT_1$ *where* $h, g_1, \dots, g_{m-p-q} \in \mathcal{E}^0_m(\Omega)$ *and where* $T_1 = dd^c g_1 \wedge \cdots \wedge dd^c g_{m-p-q} \wedge \beta^{n-m}$. *Then*

$$\int_\Omega (dd^c u_1)^p \wedge (dd^c u_2)^q \wedge T \le [\int_\Omega (dd^c u_1)^{p+q} \wedge T]^{\frac{p}{p+q}} [\int_\Omega (dd^c u_2)^{p+q} \wedge T]^{\frac{q}{p+q}}.$$

Remark 2.3. Note that for functions $u \in \mathcal{F}_m$ we have $\|u\|^m_m = \int_\Omega H_m(u)$. To see this choose $u_2 = 0$ in the infimum of the definition and hence $\|u\|^m_m \le \int_\Omega H_m(u)$. For an inequality in the other direction let $u_1, u_2 \in \mathcal{F}_m$ be any representation of $u = u_1 - u_2$.

Since $u_2 \le 0$ we have $u \ge u_1 - u_2 + 2u_2 = u_1 + u_2$. It follows from Lemma 2 in [11], we have that $\int_\Omega H_m(u) \le \int_\Omega H_m(u_1 + u_2)$ thus $\int_\Omega H_m(u) \le \|u\|_m^m$.

Next, we need some of the following results to prove the formula in Definition 2.1 is a norm on $\delta\mathcal{F}_m(\Omega)$.

Lemma 2.4. *If* $\lambda \in \mathbb{R}$ *then* $\|\lambda u\|_m = |\lambda| \|u\|_m$.

Proof. Let $\lambda > 0$. From the definition, we have

$$\|u\|_m^m = \inf_{u_1 - u_2 = u} \int_\Omega H_m(u_1 + u_2)$$

$$= \inf_{u_1 - u_2 = u} \int_\Omega H_m\left(\frac{\lambda}{\lambda}(u_1 + u_2)\right)$$

$$= \inf_{u_1 - u_2 = u} \int_\Omega \lambda^{-n} H_m(\lambda u_1 + \lambda u_2)$$

$$= \lambda^{-n} \inf_{u_1 - u_2 = u} \int_\Omega \lambda^{-n} H_m(\lambda u_1 + \lambda u_2) = \lambda^{-n} \|\lambda u\|_m^m$$

Hence $\lambda \|u\|_m = \|\lambda u\|_m$.

If $\lambda < 0$ we have $\lambda u = -\lambda(-u)$, and the same line of reasoning as above applies.

Lemma 2.5. *Suppose is a hyperconvex domain in* $\mathbb{C}^n$ *and that* $u, v \in \mathcal{F}_m(\Omega)$, *then*

$$\int_\Omega H_m(u+v) \le \left[\left(\int_\Omega H_m(u)\right)^{\frac{1}{m}} + \left(\int_\Omega H_m(v)\right)^{\frac{1}{m}}\right]^m$$

Proof. Take $h \in \mathcal{E}_m^0$ and let us consider the left hand side in the inequality above.

$$\int_\Omega -hH_m(u+v) = \sum_{j=0}^{m} \binom{m}{j} \int_\Omega -h\left(dd^c u\right)^{m-j} \wedge \left(dd^c v\right)^j$$

$$\le \sum_{j=0}^{m} \binom{m}{j} \left(\int_\Omega -hH_m(u)\right)^{m-j} \left(\int_\Omega -hH_m(v)^j\right)^{\frac{j}{m}}$$

$$= \left[\left(\int_\Omega -hH_m(u)\right)^{\frac{1}{m}} + \left(\int_\Omega -hH_m(v)\right)^{\frac{1}{m}}\right]^m$$

where the inequality comes from Lemma 2.2. Fix $w \in \Omega$, and take $h = \max(k.g\Omega - 1)$, where $g\Omega(z, \mathrm{w})$ is the pluricomplex Green function with pole at w, then $h \in \mathcal{E}_m^0$ and $h \searrow -1$ on Ω and the Lemma follows.

Now we are in a position to prove the triangle-inequality for $\delta\mathcal{F}_m$

Corollary 2.6. *Suppose* Ω *is a hyperconvex domain in* $\mathbb{C}^n$ *and that* $u, v \in \delta\mathcal{F}_m(\Omega)$, *then*

$$\|u+v\|_m \le \|u\|_m + \|v\|_m$$

Proof. Take $\varepsilon > 0$, then there is $u_i, v_i \in \mathcal{F}_m$ such that

$$\left(\int_\Omega H_m(u_1+u_2)\right)^{\frac{1}{m}} < \|u\|_m + \varepsilon$$

And

$$\left(\int_\Omega H_m(v_1+v_2)\right)^{\frac{1}{m}} < \|v\|_m + \varepsilon \ .$$

According to Lemma 2.5 we have

$$\|u\|_m + \|v\|_m - 2\varepsilon > \left(\int_\Omega H_m(u_1+u_2)\right)^{\frac{1}{m}} + \left(\int_\Omega H_m(v_1+v_2)\right)^{\frac{1}{m}}$$

$$\ge \left(\int_\Omega H_m(u_1+u_2+v_1+v_2)\right)^{\frac{1}{m}}$$

and furthermore, since $u_1+v_1-(u_2+v_2) = u - v$, u_1+v_1 and u_2+v_2 are two of the functions in the set we take infimum over we have

$$\left(\int_\Omega H_m(u_1+u_2+v_1+v_2)\right)^{\frac{1}{m}} \ge \|u+v\|_m$$

Hence $\|u+v\|_m \le \|u\|_m + \|v\|_m$.

Lemma 2.7. *If* $\|u\|_m = 0$, *then* $u = 0$.

Proof. Take $\varepsilon > 0$. Since $u_i \in \mathcal{F}_m$ such that $\int_\Omega H_m(u_1+u_2) \le \varepsilon$.

Take a sequence $\{v_j\} \in \mathcal{E}_m^0 \cap \mathcal{E}_m(\overline{\Omega})$, such that $v_j \searrow u_1+u_2$ as $j \to \infty$. Let $\phi \in \mathcal{E}_m^0$ be such that $H_m(\phi) = dV$, where dV is the Lebesgue measure. According to Lemma 3.8 in [8], with $p = m$, we have

$$\|v_j\|_{L^m}^m = \int_\Omega (-v_j)^m \, dV = \int_\Omega (-v_j)^m H_m(\phi) \le C\varepsilon^{\frac{1}{2}}$$

where $C \ge 0$ is a constant independent of j. Hence,
Hence

$$\|u\|_{L^m}^m \le \|u_1+u_2\|_{L^m}^m \le C\varepsilon^{\frac{1}{2}},$$

since $|u| = |u_1 - u_2| \le -u_1 - u_2$.

Letting $\varepsilon \searrow 0$ we get $\|u\|_{L^m} = 0$, and therefore we get that $u = 0$ almost everywhere w.r.t. *dV*. The function u is *m*-subharmonic, hence $u = 0$ everywhere on Ω.

3. One topology on $\delta\mathcal{F}_m$ space

In this section, we will give some topological properties of $\delta\mathcal{F}_m$ space. First of all, we prove that $\delta\mathcal{F}_m$ is a Banach space with the norm in the previous section.

Theorem 3.1. $(\delta\mathcal{F}_m, \|.\|_m)$ *is a Banach space.*

Proof. Lemma 2.4 and 2.7, and Corollary 2.6 shows that $(\delta\mathcal{F}_m, \|.\|_m)$ is a normed vector space. It remains to show completeness.

Suppose (u_n) is a Cauchy sequence in $\delta\mathcal{F}_m$. For each integer k there is an integer n_k such that $\|u_p - u_q\|_{L^n} \le 2^{-k}$ for $p, q > n_k$. We choose the n_k's such that $n_{k+1} > n_k$.

We have $u_{n_k} = u_{n_1} + \left(u_{n_2} - u_{n_1}\right) + \ldots + \left(u_{n_k} - u_{n_{(k-1)}}\right)$. Since $u_{n_j} \in \delta\mathcal{F}_m$ for $j = 1, \ldots, k$ we can write $u_{n_j} - u_{n_{j-1}} = \phi_j^1 - \phi_j^2$, for $\phi_j^1, \phi_j^2 \in \mathcal{F}$, where ϕ_j^1, ϕ_j^2 are chosen such that

$$\left\|u_{n_j} - u_{n_{j-1}}\right\|_m = \inf\left(\int_\Omega H_m\left(\varphi^1 + \varphi^2\right)\right)^{\frac{1}{m}} \ge \left(\int_\Omega H_m\left(\phi_j^1 + \phi_j^2\right)\right)^{\frac{1}{m}} - 2^{-j-1}$$

Then we have

$$u_{nk} = u_{n_1} + \left(\phi_2^1 - \phi_2^2\right) + \ldots + \left(\phi_k^1 - \phi_k^2\right) = u_{n_1} + \left(\phi_1^2 + \ldots + \phi_k^1\right) - \left(\phi_2^2 + \ldots + \phi_k^2\right).$$

And since $\sum_{j=2}^{k} \phi_j^1 \in \mathcal{SH}_m^-(\Omega)$ is a decreasing sequence and

$$\left(\int_\Omega H_m\left(\sum_{j=2}^{k}\phi_j^1\right)\right)^{\frac{1}{m}} \le \left(\int_\Omega H_m\left(\sum_{j=2}^{k}\phi_j^1 + \phi_j^2\right)\right)^{\frac{1}{m}} \le \sum_{j=2}^{k}\left(\int_\Omega H_m\left(\phi_j^1 + \phi_j^2\right)\right)^{\frac{1}{m}}$$

$$\le \sum_{j=2}^{k}\left(\left\|u_{n_j} - u_{n_{j-1}}\right\|_m + 2^{-j-1}\right)^{\frac{1}{m}} = \sum_{j=2}^{k}\left(2^{-j} + 2^{-j-1}\right)^{\frac{1}{m}} < \frac{1}{\sqrt[m]{2} - 1}.$$

Thus $\sum_{j=2}^{k}\phi_j^1$ is an decreasing sequence of *m*-subharmonic functions with bounded total mass, and in the same way $\sum_{j=2}^{k}\phi_j^2$ is. Therefore u_{n_k} is convergent to some $u \in \delta\mathcal{F}_m$, and since u_n is a Cauchy sequence $u_n \to u$.

Lemma 3.2. $\mathcal{F}_m$ *is closed in the topology of* $\delta\mathcal{F}_m$.

Proof. Choose a suitable sparse subsequence $\left(u_m^{'}\right)$, then $u_p = u_0 + (u_1 - u_0) + \ldots + \left(u_p - u_{p-1}\right)$, and by the exact same reasoning as in the proof of completeness for $\delta\mathcal{F}_m$ we get that $u_p \to u \in \mathcal{F}_m$.

Proposition 3.3. *The continuous functions are not dense in* $\delta\mathcal{F}_m$. *Furthermore* $\delta\mathcal{F}_m$ *is not separable.*

Proof. Let us denote the Lelong number of u at x by $v(u,x)$. The Lelong number at the origin is of course a linear functional on all of $\delta\mathcal{F}_m$, furthermore $v(\mathrm{u},0)$ is a continuous linear functional on $\delta\mathcal{F}_m$, by Proposition 3.3 in [10], we have

$$\left(2\pi v(u,x)\right)^m \le H_m(u)(\{x\}), \text{ for } u \in \mathcal{F}_m.$$

On the other hand, for all functions $u \in \mathcal{SH}_m \cap \mathcal{C}$ we have $v(u,0)=0$, thus $\log|z|$ can not be approximated by continuos functions in our topology. So that the continuous functions are not dense in $\delta\mathcal{F}_m$.

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MỘT CHUẨN *m*-HESSIAN CHO MỘT LỚP HÀM DELTA *m*-ĐIỀU HÒA DƯỚI

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Tóm tắt: *Chúng tôi xét lớp hàm $\delta\mathcal{F}_m$ bao gồm những hàm là hiệu của hai hàm thuộc $\mathcal{F}_m$ như một không gian tuyến tính, từ đó chúng tôi trang bị một chuẩn cho không gian này thông qua độ đo m-Hessian và chỉ ra với chẩn này $\delta\mathcal{F}_m$ là một không gian Banach. Hơn nữa chúng tôi cũng chỉ ra $\delta\mathcal{F}_m$ là không gian không tách được.*

Từ khóa: *Hội tụ theo dung lượng; Hàm đa điều hòa dưới; Hàm m-điều hòa dưới; Toán tử m-Hessian; Toán tử Monge–Ampère; Miền m-siêu lồi; Độ đo Radon; Độ đo dương.*