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ON THE POLYCONVOLUTION OF HARTLEY INTEGRAL TRANSFORM

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***Abstract**: In this paper, we introduce new polyconvolution related to the Hartley integral transforms and apply this polyconvolution to solve an integral equation of Toeplitz plus Hankel type and a system of two Toeplitz plus Hankel integral equations.*

***Keywords:** Toeplitz plus Hankel integral equation, Convolution, Polyconvolution, Integral tranform, Hartley transform.*

1. Introduction

In 1997, Kakichev [4] proposed a general definition of polyconvolution for $n+1$ arbitrary integral transforms $K, K_1, K_2, ..., K_n$ with the weight function $\gamma(x)$ of functions $f_1, f_2, ..., f_n$ for which the factorization property holds in the following form:

$$K\left[\overset{\gamma}{*}(f_1, f_2, ..., f_n)\right](y) = \gamma(y)(K_1 f_1)(y)(K_2 f_2)(y)...(K_n f_n)(y).$$

In this paper, the first time we construct and study a new polyconvolution for Hartley integral transforms. It's different with previous polyconvolutions, in it's factorization equality there is only Hartley integral transforms. We note that from the above factorization equality, the general definition of polyconvolution has the form

$$\overset{\gamma}{*}(f_1, f_2, ..., f_n)(x) = K^{-1}\left[\gamma(.)(K_1 f_1)(.)(K_2 f_2)(.)...(K_n f_n)(.)\right](x),$$

with K^{-1} being the inverse operator of K. Although it looks quite simple, it is not easy to have an explicit form of polyconvolution when applied to concrete integral transforms.

Furthermore, to obtain explicit formulas for polyconvolutions of different integral transforms, one should answer the question in which function space the polyconvolution live and which properties they own. Hence, we approach these goals for a new polyconvolution of Hartley integral transforms. As a by-product, we apply this new notion to solve some non-standard integral equations and a system of integral equations. We note that for such integral equation and system of integral equations, a representation of their solution in a closed form is an interesting and open problem [3, 7].

The finally, we recall well known convolution, namely convolution for the Hartley integral transform. The Hartley integral transform was introduced in [2]

$$(Hf)(x) = \int_{-\infty}^{+\infty} f(y)\,\mathrm{cas}(xy)\,dy, x \in \mathbb{R},$$

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where $\operatorname{cas}(x) = \cos x + \sin x$. The Hartley integral transform is involutive $H\,(Hf\,)(x) = f\,(x)$ and unitary $\|Hf\|_2 = \|f\|_2$. The convolution for the Hartley integral transform [5, 6]

$$\left(f \underset{H}{*} g\right)(x) = \frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{+\infty} f\,(u)\left[g\,(x+u) + g\,(x-u) + g\,(u-x) - g\,(-x-u)\right]du,$$

satisfies the factorization property

$$H\left(f \underset{H}{*} g\right)(y) = (Hf\,)(y)(Hg\,)(y).$$

This paper is organized as follow. In section 2, we introduce the polyconvolution of Hartley integral transforms. In section 3, we apply this polyconvolution to solve an integral equation of Toeplitz plus Hankel type and a system of two Toeplitz plus Hankel integral equations.

2. Polyconvolution of Hartley integral transforms

Definition 2.1. The polyconvolution for the Hartley integral transforms of the functions f, g and h is defined by

$$*(f,g,h)(x) = \frac{1}{4\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}[f\,(x+v-w) + f\,(x-v+w) - f\,(-x+v+w) + f\,(-x-v-w)\big]\,g\,(v)\,h\,(w)\,dudw,$$

$$x \in \mathbb{R} \quad (2.1).$$

Theorem 2.2.

Let f, g *and* h *be functions in* $L(\mathbb{R})$*, then the polyconvolution (2.1) for the Hartley integral transforms of the functions* f, g *and* h *belongs to* $L(\mathbb{R})$ *and the factorization property holds*

$$H\left[*(f,g,h)\right](y) = (Hf\,)(y)(Hg\,)(y)(Hh)(y), \forall y \in \mathbb{R}. \qquad (2.2)$$

Proof. First, we prove that $*(f,g,h)(x) \in L(\mathbb{R})$. Indeed,

$$\int_{-\infty}^{+\infty}\left|*(f,g,h)(x)\right|dx$$

$$\leq \frac{1}{4\pi}\int_{-\infty}^{+\infty}\left|g\,(v)\right|dv\int_{-\infty}^{+\infty}\left|h\,(w)\right|dv\int_{-\infty}^{+\infty}\left[\left|f\,(x+v-w)\right| + \left|f\,(x-v+w)\right| + \left|f\,(-x+v+w)\right| + \left|f\,(-x-v-w)\right|\right]dx$$

. It is easy to see that

$$\int_{-\infty}^{+\infty}\left[\left|f\,(x+v-w)\right| + \left|f\,(x-v+w)\right| + \left|f\,(-x+v+w)\right| + \left|f\,(-x-v-w)\right|\right]dx = 4\int_{-\infty}^{+\infty}\left|f\,(t)\right|dt.$$

Hence,

$$\int_{-\infty}^{+\infty}\left|*(f,g,h)(x)\right|dx \leq \frac{1}{\pi}\int_{-\infty}^{+\infty}\left|g\,(v)\right|dv\int_{-\infty}^{+\infty}\left|h\,(w)\right|dw\int_{-\infty}^{+\infty}\left|f\,(t)\right|dt < +\infty.$$

Therefore, $*(f,g,h)(x)$ belongs to $L(\mathbb{R})$.

Now we prove the factorization property (2.2). Since

$$(Hf\,)(y)(Hg\,)(y)(Hh)(y) = \frac{1}{2\pi}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\operatorname{cas}(yu).\operatorname{cas}(yv).\operatorname{cas}(yw).f\,(u)\,g\,(v)\,h\,(w)\,dudvdw,$$

and

$$\mathrm{cas}(yu).\mathrm{cas}(yv).\mathrm{cas}(yw)$$

$$= \frac{1}{2}\left[\mathrm{cas}\, y(u - v + w) + \mathrm{cas}\, y(u + v - w) + \mathrm{cas}\, y(-u + v + w) - \mathrm{cas}\, y(-u - v - w)\right],$$

We oblain

$$(Hf)(y)(Hg)(y)(Hh)(y) = \frac{1}{4\pi\sqrt{2\pi}} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left[\mathrm{cas}\, y(u - v + w) + \mathrm{cas}\, y(u + v - w) + \mathrm{cas}\, y(-u + v + w)\right.$$

$$\left. - \mathrm{cas}\, y(-u - v - w)\right] f(u) g(v) h(w)\, dudvdw = \frac{1}{4\pi\sqrt{2\pi}} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathrm{cas}(yt)\left[f(t + v - w) + f(t - v + w)\right.$$

$$\left. - f(-t + v + w) + f(-t - v - w)\right] g(v) h(w)\, dtdvdw$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} *(f, g, h)(y)\, \mathrm{cas}\, yt dt = H\left[*(f, g, h)\right](y), \forall y \in \mathbb{R}.$$

The proof is completed.

Theorem 2.3. (Titchmarch-type Theorem)

Let $f, g, h \in L(\mathbb{R})$. *If* $\forall x \in \mathbb{R}, *(f, g, h)(x) \equiv 0$, *then either* $f(x) = 0$, *or* $g(x) = 0$, *or* $h(x) = 0, \forall x \in \mathbb{R}$.

Proof. The hypothesis $*(f, g, h)(x) \equiv 0$ implies that

$$H\left[*(f, g, h)\right](y) = 0, \forall y \in \mathbb{R}.$$

Due to Theorem 2.2 we have

$$(Hf)(y)(Hg)(y)(Hh)(y) = 0, \forall y \in \mathbb{R}. \tag{2.3}$$

As $(Hf)(y), (Hg)(y), (Hh)(y)$ are analytic functions for all y in $\mathbb{R}$, so from (2.3) we have $(Hf) = 0, \forall y \in \mathbb{R}$, or $(Hg) = 0, \forall y \in \mathbb{R}$, or $(Hh) = 0, \forall y \in \mathbb{R}$.

It follows that either $f(x) = 0, \forall x \in \mathbb{R}$, or $g(x) = 0, \forall x \in \mathbb{R}$, or $h(x) = 0, \forall x \in \mathbb{R}$.

The theorem is proved. □

In the sequel, for simplicity, we define the norm in the space $L(\mathbb{R})$ by

$$\|f\| = \frac{1}{\sqrt[3]{\pi}} \int_{-\infty}^{+\infty} |f(x)|\, dx .$$

Theorem 2.4. *If f,g,h belong to* $L(\mathbb{R})$, *then the following inequality holds*

$$\|*(f, g, h)\| \le \|f\|.\|g\|.\|h\|.$$

Proof. From the proof of Theorem 2.2, we obtain

$$\int_{-\infty}^{+\infty} |*(f, g, h)(x)|\, dx \le \frac{1}{\pi} \int_{-\infty}^{+\infty} |f(t)|\, dt \int_{-\infty}^{+\infty} |g(v)|\, dv \int_{-\infty}^{+\infty} |h(w)|\, dw$$

$$= \frac{1}{\sqrt[3]{\pi}} \int_{-\infty}^{+\infty} |f(t)|\, dt . \frac{1}{\sqrt[3]{\pi}} \int_{-\infty}^{+\infty} |g(v)|\, dv . \frac{1}{\sqrt[3]{\pi}} \int_{-\infty}^{+\infty} |h(w)|\, dw .$$

Thus $\|*(f, g, h)\| \le \|f\|.\|g\|.\|h\|$.

The proof is completed. □

Theorem 2.5. *In the space* $L(\mathbb{R})$*, the polyconvolution for the Hartley integral transforms is commutative, associative and distributive.*

Proof. We prove that the polyconvolution for the Hartley integral transforms is commutative, i.e.,

$$*(f,g,h) = *(f,h,g) = *(g,f,h) = *(g,h,f) = *(h,f,g) = *(h,g,f).$$

Indeed

$$H\left[*(f,g,h)\right](y) = (Hf)(y).(Hg)(y).(Hh)(y)$$

$$= (Hf)(y).(Hh)(y).(Hg)(y) = H\left[*(f,h,g)\right](y), \forall y \in \mathbb{R}.$$

Implies that $*(f,g,h) = *(f,h,g)$.

The following equalities are similarly proved. The associative, distributive properties are similarly proved.

3. Application to solve an integral equation and a system of integral equations

First, we consider the integral equation

$$f(x) + \frac{1}{4\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left[f(x+v-w) + f(x-v+w) + f(-x+v+w) - f(-x-v-w)\right]g(v)h(w)\,dvdw$$
$$= k(x), x \in \mathbb{R}. \quad (3.1)$$

Here g, h and k are given functions of $L(\mathbb{R})$, f is unknown function.

Theorem 3.1. *Under the condition* $1 + (Hg)(y)(Hh)(y) \neq 0, \forall y \in \mathbb{R}$, *there exists a unique solution*
in $L(\mathbb{R})$ *of (3.1) which is defined by*

$$f = k - \left(k \underset{H}{*} l\right).$$

Here, $l \in L(\mathbb{R})$ *and it is determined by the equation*

$$(Hl)(y) = \frac{H\left(g \underset{H}{*} h\right)(y)}{1 + H\left(g \underset{H}{*} h\right)(y)}.$$

Proof. The equation (3.1) can be rewriten in the form

$$f(x) + \left[*(f,g,h)\right](x) = k(x).$$

Due to Theorem 2.2 $(Hf)(y) + (Hf)(y)(Hg)(y)(Hh)(y) = (Hk)(y), \forall y \in \mathbb{R}$. It follows that

$$(Hf)(y)\left[1 + (Hg)(y)(Hh)(y)\right] = (Hk)(y).$$

Since $1 + (Hg)(y)(Hh)(y) \neq 0$,

$$(Hf)(y) = (Hk)(y).\frac{1}{1 + (Hg)(y)(Hh)(y)}.$$

Therefore,

$$(Hf)(y) = (Hk)(y).\left[1 - \frac{(Hg)(y)(Hh)(y)}{1 + (Hg)(y)(Hh)(y)}\right] = (Hk)(y)\left[1 - \frac{H\left(g \underset{H}{*} h\right)(y)}{1 + H\left(g \underset{H}{*} h\right)(y)}\right].$$

Due to Wiener-Levy's theorem in [1], there exists a function $l \in L(\mathbb{R})$ such that

$$(Hl)(y) = \frac{H\left(g \underset{H}{*} h\right)(y)}{1 + H\left(g \underset{H}{*} h\right)(y)}.$$

It follows that

$$(Hf)(y) = (Hk)(y) - H\left(k \underset{H}{*} l\right)(y).$$

Thus,

$$f = k - \left(k \underset{H}{*} l\right).$$

It is easy to see that $f \in L(\mathbb{R})$. The theorem is proved.

Next, we consider the system of integral equations

$$f(x) + \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[g(x+v-w) + g(x-v+w) + g(-x+v+w) + g(-x-v-w)\right]\varphi(v)\psi(w)\,dvdw = h(x)$$

$$\frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{\infty} f(v)\left[p(x+v) + p(x-v) + p(v-x) - p(-x-v)\right]dv + g(x) = k(x), x \in \mathbb{R}.$$

(3.2)

Where φ, ψ, p, h and k are given functions in $L(\mathbb{R})$, and f and g are the unknown functions.

Theorem 3.2. *Under the condition* $1 - H\left[*(p,\varphi,\psi)\right](y) \neq 0, \forall y \in \mathbb{R}$ *, there exists a unique solution in* $L(\mathbb{R})$ *of (3.2) which is defined by*

$$f(x) = \left(h \underset{H}{*} k\right)(x) - \left[*(k,\varphi,\psi) \underset{H}{*} l\right](x) + h(x) - *(k,\varphi,\psi)(x) \in L(\mathbb{R}),$$

$$g(x) = k(x) - \left(h \underset{H}{*} p\right)(x) + \left(k \underset{H}{*} l\right)(x) - \left[\left(k \underset{H}{*} p\right) \underset{H}{*} l\right](x) \in L(\mathbb{R}).$$

Here $l \in L(\mathbb{R})$ *and defined by the equations*

$$(Hl)(y) = \frac{H\left[*(p,\varphi,\psi)\right](y)}{1 - H\left[*(p,\varphi,\psi)\right](y)}.$$

Proof. System (3.2) can be written in the form

$$f(x) + *(g,\varphi,\psi)(x) = h(x),$$

$$\left(f \underset{H}{*} p\right)(x) + g(x) = k(x), x \in \mathbb{R}.$$

Using the factorization property of the polyconvolution (2.1) and the convolution (1.2) we obtain the linear system of algebraic equations with respectively to $(Hf)(y)$ and $(Hg)(y)$

$$(Hf)(y) + (Hg)(y).(H\varphi)(y)(H\psi)(y) = (Hh)(y),$$

$$(Hf)(y)(Hp)(y) + (Hg)(y) = (Hk)(y), y \in \mathbb{R}.$$

Formally, we have

$$\Delta = \begin{vmatrix} 1 & (H\varphi)(y)(H\psi)(y) \\ (Hp)(y) & 1 \end{vmatrix} = 1 - (Hp)(y)(H\varphi)(y)(H\psi)(y) = 1 - H\left[*(p,\varphi,\psi)\right](y).$$

$$\Delta_1 = \begin{vmatrix} (Hh)(y) & (H\varphi)(y)(H\psi)(y) \\ (Hk)(y) & 1 \end{vmatrix} = (Hh)(y) - (Hk)(y)(H\varphi)(y)(H\psi)(y)$$

$$= (Hh)(y) - H\left[*(k,\varphi,\psi)\right](y).$$

$$\Delta_2 = \begin{vmatrix} 1 & (Hh)(y) \\ (Hp)(y) & (Hk)(y) \end{vmatrix} = (Hk)(y) - H\left(h \underset{H}{*} p\right)(y).$$

Note that $1 - H\left[*(k,\varphi,\psi)\right](y) \neq 0$,

$$(Hf)(y) = \left\{(Hh)(y) - H\left[*(k,\varphi,\psi)\right](y)\right\}.\frac{1}{1 - H\left[*(p,\varphi,\psi)\right](y)}$$

$$= \left\{(Hh)(y) - H\left[*(k,\varphi,\psi)\right](y)\right\}.\left\{1 + \frac{H\left[*(p,\varphi,\psi)\right](y)}{1 - H\left[*(p,\varphi,\psi)\right](y)}\right\}.$$

So according to Wiener-Levy's theorem [1], there exists a function $l \in L(\mathbb{R})$ such that

$$(Hl)(y) = \frac{H\left[*(p,\varphi,\psi)\right](y)}{1 - H\left[*(p,\varphi,\psi)\right](y)}.$$

It follows that

$$(Hf)(y) = \left\{(Hh)(y) - H\left[*(k,\varphi,\psi)\right](y)\right\}.\left\{1 + (Hl)(y)\right\}$$

$$= H\left(h \underset{H}{*} l\right)(y) - H\left\{\left[*(k,\varphi,\psi)\right] \underset{H}{*} l\right\}(y) + (Hh)(y) - H\left[*(k,\varphi,\psi)\right](y).$$

Thus,

$$f(x) = \left(h \underset{H}{*} l\right)(x) - \left[*(k,\varphi,\psi) \underset{H}{*} l\right](x) + h(x) - \left[*(k,\varphi,\psi)\right](x) \in L(\mathbb{R}).$$

Similarly we obtain

$$(Hg)(y) = \left\{(Hk)(y) - H\left(h \underset{H}{*} p\right)(y)\right\}\left\{1 + (Hl)(y)\right\}$$

$$= (Hk)(y) - H\left(h \underset{H}{*} p\right)(y) + H\left(k \underset{H}{*} l\right)(y) - H\left[\left(k \underset{H}{*} p\right) \underset{H}{*} l\right](y).$$

It follows that

$$g(x) = k(x) - \left(h \underset{H}{*} p\right)(x) + \left(k \underset{H}{*} l\right)(x) - \left[\left(k \underset{H}{*} p\right) \underset{H}{*} l\right](x), \in L(\mathbb{R}).$$

The proof is completed.

4. Conclusion

In this paper, we introduce a new polyconvolution for Hartley integral transforms in the form

$$*(f,g,h)(x) = \frac{1}{4\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left[f(x+v-w) + f(x-v+w) - f(-x+v+w) + f(-x-v-w)\right]g(v)h(w)\,du\,dw.$$

We apply this new notion to solve some non-standard integral equations and a system of integral equations. We note that for such integral equation and system of integral equations, a representation of their solution in a closed form is an interesting and open problem.

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ĐA CHẬP ĐỐI VỚI PHÉP BIẾN ĐỔI TÍCH PHÂN HARTLEY

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Trường Đại học Điện lực

***Tóm tắt**: Trong bài báo này, chúng tôi giới thiệu đa chập đối với phép biến đổi tích phân Hartley và áp dụng đa chập này vào giải phương trình và hệ phương trình tích phân dạng Toeplitz- Hankel.*

***Từ khóa:** Phương trình tích phân Toeplitz-Hankel, Tích chập, Đa chập, Phép biến đổi tích phân, Phép biến đổi Hartley.*