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LYAPUNOV STABILITY OF SOLUTION FOR NONLINEAR ITÔ-TYPE SYSTEMS

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Abstract: *In this work we study the stability of solution for Itô nonlinear stochastic discrete-time systems. First, we introduce several notions on stability of solutions. Second, by using Lyapunov functionals method we prove some results about stochastic stability of solution.*

Keywords: *Stochastic stability, Itô discrete-time systems, Lyapunov functionals method.*

1. Introduction

During the 18th century, astronomers and mathematicians made great efforts to show that the observed deviations of planets and satellites from fixed elliptical orbits were in agreement with Newton's principle of universal gravitation, provided that due account was taken of the disturbing forces exerted by the bodies on one another. The deviations are of two kinds: first, oscillatory motions with relatively short periods, i.e. periods of the order of a few years, and second, residual slow changes in the ellipse parameters, which changes may be non-oscillatory or may be oscillatory with very long periods, perhaps of the order of tens of thousands of years. The first kind are known as periodic inequalities, and may be accounted for as the response of a body to the periodic forces exerted on it by its neighbours' continual tracing of their orbits. The second kind are called secular inequalities, and for the solar system the question arises as to whether the secular inequalities will build up over the millennia and destroy the system. In 1892, Lyapunov introduced the concept of stability of dynamic systems and created a very powerful tool known as the Lyapunov method in the study of stability. It can be found that the Lyapunov method has been developed and applied to investigate stochastic stability of the Itô-type systems, and many important classical results on deterministic differential equations have been generalized to the stochastic Itô systems; we refer the reader to Arnold [1], Friedman [2], Has'minskii[4], Kushner [5], Kolmanovskii and Myshkis [6]. Stability is the first of all the considered problems in the system analysis and synthesis of modern control theory, which plays an essential role in dealing with infinite-horizon linear-quadratic regulator, H_2/H_∞ robust optimal control, and other control problems; see [3,7,8].

Compared with the plenty of results of the continuoustime Itô systems, few results have been obtained on the stability of discrete-time nonlinear stochastic systems.

In this work, we study some types of stabilities in probability for the n-dimensional stochastic discrete-time system

$$\begin{cases} u(t+1) = f(u(t), w(t), t), t > 0 \\ u(0) = u_0 \end{cases} \quad (1.1)$$

where $u_0 \in \mathbb{R}^n$ is a constant vector. For any given initial value $u(0) = u_0 \in \mathbb{R}^n$, $w(t)$ is a one-dimensional stochastic process defined on the complete probability space (Ω, F, P). We assume that $f(0, w(t), t) \equiv 0$ for all $t \in I := \{k : k \in \mathbb{Z}^+\}$, so (1.1) has the solution $u(t) \equiv 0$ corresponding to the initial value $u(0) = 0$. This solution is called the trivial solution or the equilibrium position.

For convenience, we adopt the following notations:

$$K := \{\varphi \in C([0,+\infty);[0,+\infty)):$$
$$\text{strictly increasing and } \varphi(0)=0\}$$

$D_r := \{x \in \mathbb{R}^n : |x| < r\}$ for $r > 0$;

$C^2(U)$: the class of functions twice continuously differential on U;

$a \wedge b$: the minimum of a and b.

2. Preliminaries

In this section, we recall some results and notions related to stabilities in probability of Itô-type system (1.1).

Definition 2.1. A Lyapunov function for an autonomous dynamical system

$$\begin{cases} y : \mathbb{R}^n & \to \mathbb{R}^n \\ \dot{y} & = g(y) \end{cases}$$

with an equilibrium point at $y = 0$ is a scalar function $V : \mathbb{R}^n \to \mathbb{R}$ that is continuous, has continuous first derivatives, is locally positive-definite, and for which $-\nabla V.g$ is also locally positive definite. The condition that $-\nabla V.g$ is locally positive definite is sometimes stated as $\nabla V.g$ is locally negative definite.

Definition 2.2. The trivial solution of (1.1) is said to be stochastically stable or stable in probability if, for every $\epsilon > 0$ and $h > 0$, there exists $\sigma = \sigma(\epsilon, h) > 0$, such that

$$P\{|u(t)| < h\} \geq 1 - \epsilon, t \geq 0,$$

when $|u| < \sigma$. Otherwise, it is said to be stochastically unstable.

Proposition 2.3. If there exists a positive definite function $V(u) \in C^2(D_r)$, $D_r := \{x \in \mathbb{R}^n : |x| < r\}$ such that:

$$E[\Delta V(u(t))] \leq 0,$$

for all $u(t) \in D_r$, then the trivial solution of (1.1) is stochastically stable in probability.

Proof.

By the definition of $V(u)$, we have $V(0) = 0$ and there exists $\varphi \in K$, such that:

$$V(u(t)) \geq \varphi(|u|), \forall u \in D_r.$$

For every $\epsilon \in (0,1)$ and $h > 0$, no loss of generality, we assume that $h < r$. Because $V(u)$ is continuous, we can find that $\sigma = \sigma(\epsilon, h) > 0$, such that:

$$V(u(t)) \leq \epsilon\varphi(h), \forall u \in D_\sigma. \quad (2.1)$$

Clearly $\sigma < h$. We fix the initial value $u_0 \in D_\sigma$. Let μ be the first exit time of $u(t)$ from D_h; that is: $\mu = \inf\{t \geq 0 : u(t) \notin D_h\}$. Let $\tau = \mu \wedge t$, for any $t \geq 0$, we have:

$$\begin{aligned} V(u(\mu \wedge t)) - V(u_0) &= V(u(\tau)) - V(u(\tau - 1)) \\ &+ V(u(\tau - 1)) - V(u(\tau - 2)) + \dots \\ &+ V(u(t_0 + 1)) - V(u_0) \\ &= \sum_{t=t_0}^{\tau-1} \Delta V(u(t)). \end{aligned}$$

Taking the expectation on both sides, it is easy to see that

$$EV(u(\mu \wedge t)) \leq V(u_0). \quad (2.2)$$

If $\mu \leq t$ and we note that

$$|u(\mu \wedge t)| = |u(\mu)| = h,$$

then:

$$\begin{aligned} \varphi(h) P\{\mu \leq t\} &\leq E[I_{\{\mu \leq t\}} V(u(\mu))] \\ &\leq EV(u(\mu \wedge t)). \end{aligned}$$

From (2.1) and (2.2), we have: $P\{\mu \leq t\} \leq \epsilon$. Let $t \to +\infty$, then we have $P\{\mu < \infty\} \leq \epsilon$, it means that:

$$P\{|u(t)| < h\} \geq 1 - \epsilon, t \geq 0.$$

Therefore, the trivial solution of (1.1) is stochastically stable.

Definition 2.4. The trivial solution of (1.1) is said to be stochastically asymptotically stable in probability if it is stochastically stable, and for every $\epsilon > 0$, there exists $\sigma = \sigma(\epsilon) > 0$, such that $P\{\lim_{t\to\infty} u(t) = 0\} \geq 1 - \epsilon$, when $|u| < \sigma$.

Definition 2.5. The trivial solution of (1.1) is said to be stochastically asymptotically stable in the large in probability if it is stochastically stable, and $P\{\lim_{t\to\infty} u(t) = 0\} = 1$, for all $u_0 \in \mathbb{R}^n$.

3. Main results

In this section, we prove some results about stochastically stable of system (1.1) by appling Lyapunov functionals method.

Theorem 3.1. If there exists a function $\varphi \in K$ and a positive definite function $V(u) \in C^2(D_r)$, such that

$$E[\Delta V(u(t))] \le -E\varphi(|u(t)|),$$

for all $u(t) \in D_r$, then the trivial solution of (1.1) is stochastically asymptotically stable in probability.

Proof.

From Proposition 2.3, we have that the trivial solution of (1.1) is stochastically stable. Fix $\epsilon \in (0,1)$ arbitrarily; then there exists $\sigma_0 = \sigma_0(\epsilon) > 0$, such that:

$$P\{|u(t)| < \frac{r}{2}\} \ge 1 - \frac{\epsilon}{4}, \qquad (3.1)$$

here $u_0 \in D_{\sigma_0}$.

Fix $u_0 \in D_{\sigma_0}$. By the assumptions on function $V(u)$, we see that $V(0) = 0$ and there exist two functions $\varphi_1, \varphi \in K$, such that:

$$\varphi_1(|u|) \le V(u), E[\Delta V(u(t))]$$
$$\le -E\varphi(|u(t)|), \forall u \in D_r.$$

Let $0 < \beta < |u_0|$ and choose $0 < \alpha < \beta$, $0 < \eta < \alpha$ small enough; because $V(u)$ is continous, we see that $0 < \sigma = \sigma(\epsilon) < \sigma_0$, such that:

$$V(u) \le \frac{\epsilon}{4}\varphi(\eta), \forall \in D_\sigma. \qquad (3.2)$$

Define the stoping times

$$\mu_\alpha = \inf\{t \ge 0 : |u(t)| \le \alpha\},$$

$$\mu_r = \inf\left\{t \ge 0 : |u(t)| \ge \frac{r}{2}\right\}.$$

Choose θ sufficiently large, such that: $P\{\mu_\alpha < \theta\} \ge 1 - \frac{\epsilon}{4}$. Let $\tau = \mu_\alpha \wedge \mu_r \wedge t$, for all $t \ge 0$, we have:

$$V(u(\tau)) - V(u_0) = V(u(\tau)) - V(u(\tau-1))n$$
$$+ V(u(\tau-1)) - V(u(\tau-2))n$$
$$+ \dots + V(u(t_0+1)) - V(u_0)$$
$$= \sum_{t=0}^{\tau-1} \Delta V(u(t)) \ge 0.$$

Taking the expectation on both sides, we have: $0 \le EV(u(\tau)) \le V(u_0) - \varphi(\alpha)(\tau)$.

Hence,

$$\frac{V(u_0)}{\varphi(\alpha)} \ge E(\mu_\alpha \wedge \mu_r \wedge t)$$
$$= E(\tau) \ge (t)P\{\mu_\alpha \wedge \mu_r \ge t\}.$$

This means that: $P\{\mu_\alpha \wedge \mu_r < \infty\} = 1$. By (3.1) we have $\{\mu_r < \infty\} \le \frac{\epsilon}{4}$. So

$$P\{\mu_\alpha < \infty\} + \frac{\epsilon}{4} \ge P\{\mu_\alpha < \infty\} + P\{\mu_r < \infty\}$$
$$\ge P\{\mu_\alpha \wedge \mu_r < \infty\} = 1,$$

it lead to: $1 - \frac{\epsilon}{4} \le P\{\mu_\alpha < \infty\}$.

Hence,

$$P\{\mu_\alpha < \mu_r \wedge \theta\} \ge P(\{\mu_\alpha < \theta\} \cap \{\mu_r = \infty\})$$
$$\ge P\{\mu_\alpha < \theta\} - P\{\mu_r < \infty\}$$
$$\ge 1 - \frac{3}{4}\epsilon. \qquad (3.3)$$

Define the two stopping times

$$\sigma = \begin{cases} \mu_\alpha & \text{if} \quad \mu_\alpha < \mu_r \wedge \theta \\ \infty & \text{otherwise} \end{cases}$$

$$\mu_\beta = \inf\{t > \sigma : |x(t)| \ge \beta\}.$$

We show that, for $t \ge \theta$,

$$EV(u(\sigma \wedge t)) \ge EV(u(\mu_\beta \wedge t)).$$

If $\mu_\alpha \ge \mu_r \wedge \theta$, note that

$$|u(\mu_\beta \wedge t)| = |u(\sigma \wedge t)| = |u(t)| = \eta,$$

then

$$E\left[I_{\{\mu_\alpha < \mu_r \wedge \theta\}} V(u(\mu_\alpha))\right] \ge E\left[I_{\{\mu_\alpha < \mu_r \wedge \theta\}} V(u(\mu_\beta \wedge t))\right]$$

Due to (3.1) and

$$\{\mu_\alpha < \mu_r \wedge \theta\} \supset \{\mu_\beta \le t\},$$

we have

$$\varphi_1(\eta)P\{\mu_\beta \le t\} \le E\left[I_{\{\mu_\beta \wedge t\}} V(u(\mu_\beta \wedge t))\right]$$
$$\le EV(u(\mu_\beta \wedge t))$$

Combine with (3.2), we have: $\frac{\epsilon}{4} \ge P\{\mu_\beta \le t\}$. Let $t \to \infty$, one gets:

$$\frac{\epsilon}{4} \ge P\{\mu_\beta \le \infty\}.$$

By (3.3), we have:

$$1 - \epsilon \le P\{\mu_\alpha < \mu_h \wedge \theta\} - P\{\mu_\beta < \infty\}$$
$$\le P\{\sigma < \infty, \mu_\beta = \infty\}$$

This means that

$$P\{\limsup_{t \to \infty} |u(t)| \le \beta\} \ge 1 - \epsilon.$$

Because β is arbitrary, then we have $P\{\lim_{t\to\infty} u(t)=0\}\geq 1-\epsilon$. The proof is complete.

Theorem 3.2. If there exists a function $\varphi \in K$ and a positive definite radially unbounded function $V(u)\in C^2(D_r)$, such that

$$E[\Delta V(u(t))] \leq -E\varphi(|u(t)|),$$

for all $u(t)\in D_r$, then the trivial solution of (1.1) is stochastically asymptotically stable in the large in probability.

Proof.

From Proposition 2.3, we have that the trivial solution of (1.1) is stochastically stable. Let $\epsilon \in (0,1)$ arbitrary and fix u_0. Because $V(u)$ is radially unbounded, then we can choose $r>|u_0|$ large enough, such that:

$$\inf_{|u|\geq r, t\geq t_0} V(u) \geq \frac{4V(u_0)}{\epsilon}. \quad (3.4)$$

Define the stopping time

$$\mu_r = \inf\{t\geq 0 : |u(t)|\geq r\}.$$

Taking the expectation on both sides, we see that for all $t\geq 0$,

$$V(u_0)\geq EV(u(\mu_r \wedge t)) \quad (3.5)$$

From (3.4), we have:

$$V(u_0)\geq EV(u(\mu_r\wedge t)) \geq \frac{4V(u_0)}{\epsilon} P\{\mu_r \leq t\}.$$

Combine with (3.5), we obtain:

$$P\{\mu_r \leq t\} \leq \frac{\epsilon}{4}.$$

Let $t\to\infty$; we have $P\{\mu_r\leq \infty\}\leq \frac{\epsilon}{4}$. It means that $P\{|u(t)|\leq r\}\geq 1-\frac{\epsilon}{4}, \forall t\geq 0.$ Similar to the proof of Theorem 3.1, we can obtain $P\{\lim_{t\to+\infty} u(t)=0\}\geq 1-\epsilon.$

This implies that $P\{\lim_{t\to+\infty} u(t)=0\}\geq 1$. The proof is complete.

4. Example

In this section, we give an application to the abstract results. Let $w(t)$ be a one-dimensional stochastic process defined on the complete probability space (Ω, F, P), such that $Ew(t)=0$ and $E[w(t)w(s)]=\delta_{st}$, here δ_{st} is Kronecker delta.

Consider the following stochastic difference equation:

$$\begin{aligned} u(t+1) &= [M(t)+N(t)w(t)]u(t) \\ &= K(t,w(t))u(t), \end{aligned} \quad (4.1)$$

with $M(t), N(t)$, and

$$K(t,w(t)) = M(t)+N(t)w(t) = (k_{i,j}(t,w(t)))$$

are all 2×2 matrix-valued functions defined on $t=0,1,\dots$ and $u(0)=u_0\in \mathbb{R}^n$. Assume that

$$\max_{i=1,2} E\left(\sum_{j=1}^{2} |k_{i,j}(t,\mathrm{w}(t))|^2\right) < \frac{1}{2},$$

for all $u(t)\in\mathbb{R}^2$.

We define the Lyapunov function

$$V(u) = \max_{i=1,2}\{|u_i|^2\}.$$

It is positive definite and radially unbounded. Moreover,

$$\begin{aligned} EV(u(t+1)) &= \max_{i=1,2} E\left(|\sum_{j=1}^{2} k_{i,j}(t,w(t))u_j(t)|^2\right) \\ &\leq \max_{i=1,2} E\left(\sum_{j=1}^{2}|k_{i,j}(t,w(t))|^2 \sum_{j=1}^{2}|u_j(t)|^2\right) \\ &\leq \max_{i=1,2} E\left(\sum_{j=1}^{2}|k_{i,j}(t,w(t))|^2\right) \\ &\quad \times \max_{j=1,2} E\left(|u_j(t)|^2\right) \\ &< \max_{j=1,2} E\left(|u_j(t)|^2\right) = EV(u(t)). \end{aligned}$$

That is, $E[\Delta V(u(t))]<0$. By Theorem 3.2, the trivial solution is stochastically asymptotically stable in the large in probability.

5. Conclusion

In this paper we construct some notions and apply Lyapunov functionals method to study the stability of solution for nonlinear Itô stochastic discrete-time systems (1.1). After that, we give an application to the abstract results.

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TÍNH ỔN ĐỊNH LYAPUNOV CỦA NGHIỆM ĐỐI VỚI HỆ ITÔ PHI TUYẾN

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Tóm tắt: *Trong công trình này chúng tôi nghiên cứu tính ổn định nghiệm của hệ ngẫu nhiên phi tuyến thời gian rời rạc. Trước tiên, chúng tôi giới thiệu một số định nghĩa liên quan đến tính ổn định của nghiệm. Sử dụng phương pháp hàm Lyapunov để chứng minh một số kết quả về tính ổn định ngẫu nhiên của hệ ngẫu nhiên phi tuyến thời gian rời rạc.*

Từ khóa: *Ổn định ngẫu nhiên, hệ Itô thời gian rời rạc, phương pháp hàm Lyapunov.*

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