OUTER APPROXIMATION ALGORITHMS FOR EQUILIBRIUMS

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Abstract: *Equilibrium problems appear frequently in many practical problems arising from, for instance, physics, engineering, game theory, transportation, economics and network. They have become an attractive field for many researchers in both theory and applications. The paper concerns with some methods for finding an approximate solution of equilibrium problems.*

Using theoretical development methods, we present two new outer approximation algorithms for solving equilibrium problems on a convex subset C, where the underlying function is Lipschitz-type continuous and pseudomonotone. The algorithm is to define proximal point subproblems or projector point subproblems on the convex domains $C_k \supseteq C$, $C_{k+1} \subset C_k$, $\cap_{k=0}^{\infty} C_k = C$, $k = 0,1,...$ *The strongly convergent theorems are established under standard assumptions imposed on cost mappings.*

Keywords: *Equilibrium problem; Proximal point problem; Pseudonomotone; Lipschitz-type continuous; Outer approximation algorithm.*

1. Introduction

Let C be a nonempty closed convex subset of $\mathbb{R}^n$. We present outer approximation algorithms for finding a solution of equilibrium problems (shortly EP):

Find $x^* \in C$ such that $f(x^*, y) \geq 0, \ \forall y \in C.$

(EP)

The area of equilibrium problems has attracted attention of researchers from applications in physics, engineering, game theory, transportation, economics and network ([1-9]). problems are models whose formulation includes optimization, variational inequalities, multi-objective optimization problems, fixed point problems, saddle point problems, Nash equilibria and complementarity problems as particular cases ([5, 7, 11]). Recently, researchers have presented some methods to solve EP*(f, C)* with a monotone bifunction *f*.

In [5], the authors used projection methods. The interior quadratic regularization methods is presented in [1,2]. In [4,12], the interior quadratic regularization technique hasbeen used to develop proximal iterative algorithm for variational inequalities. In [3], P.N. Anh, T. Kuno used a cutting hyperplane method for pseudomonotone equilibrium problems. In this paper we present two new outer approximation algorithms for solving equilibrium problems on a convex subset C. The first algorithm is to define proximal point subproblems on the convex domains $C_k \supseteq C$, $\cap_{k=0}^{\infty} C_k = C$, $C_{k+1} \subset C_k$, $k = 0,1,...$, which forms a generalized iteration scheme for finding a global equilibrium point. The second algorithm is to define projector point subproblems on the convex domain, which forms a generalized iteration scheme for finding two projector points on the C_k.

The paper is organized as follows. In sections 2 we review some definitions and basic results that will be used in our subsequent analysis. Section 3, will present two outer approximation algorithms for solving equilibrium problems on a convex subset C and prove convergence of the algorithms.

2. Preliminaries

This section contains some definitions and basic results that will be used in our subsequent analysis. We first state the formal definition of some classes of functions that play an essential role in this paper.

Let C be a nonempty closed convex subset of a real space $\mathbb{R}^n$. A bifunction $f: C\times C \to \mathbb{R}^n$ is called to be

(i) β -strongly monotone, if

$$f(x,y)+f(y,x)\le -\beta \|x-y\|^2 \ \forall x,y\in C;$$

(ii) monotone, if

$$f(x,y)+f(y,x)\le 0 \ \forall x,y\in C;$$

(iii) pseudonomotone, if $f(x,y)\ge 0$ thì $f(y,x)\ge 0,\forall x,y\in C$

(iv) Lipschitz-type continuous with constants $c_1>0$ and $c_2>0$ ([8]), if

$$f(x,y)+f(y,z)\ge f(x,z)-c_1\|x-y\|^2 -c_2\|y-z\|^2 \ \forall x,y,z\in C.$$

Lemma 2.1. ([13]) *Let $\{a^k\}$, $\{b^k\}$ and $\{c^k\}$ be the three positive real numerical satisfying the following conditions:*

$$a_{n+1}\le (1+b_n)a_n+c_n, \forall n\ge 0, \sum_{n=0}^{\infty} b_n<+\infty,$$

$\sum_{n=0}^{\infty} c_n<+\infty.$ *Then,* $\lim a_n$ *exists.*

3. Outer approximation algorithms

In this section, we present new outer approximation algorithms for solving equilibrium problems on a convex subset C. Its convergence analysis is postponed. We first state the assumptions that we will assume to hold through the rest of this paper.

Assumption:

(A1) f is pseumonotone on $\mathbb{R}^n$

(A2) f satisfies Lipschitz-type condition on $\mathbb{R}^n$ with two constants c_1 and c_2.

(A3) $f(x,.)$ is convex and subdifferentiable on $\mathbb{R}^n$ for every fixed $x\in\mathbb{R}^n$.

(A4) f is lower semicontinuous on $\mathbb{R}^n\times\mathbb{R}^n$.

3.1 Algorithm 1

Initialization: Choose the sequence C_k such that $C_k\supseteq C$, $\cap_{k=0}^{\infty} C_k=C$, $C_{k+1}\subset C_k$, $k=0,1,...$, $x^0\in \text{int}C$, positive sequences λ_k ***Iteration k:***

$$\begin{cases}\text{Find } x^{k+1}\in C_{k+1}\\ \lambda_k[f(x^k,x)-f(x^k,x^{k+1})]+f(x^k,x)\\ -\langle x^{k+1}-x^k, x^{k+1}-x\rangle \ge 0, \forall x\in C_{k+1}. \end{cases} \quad (1)$$

Lemma 3.1.1. *We fix sequences $\{C_k\}$ of closed convex subsets of $\mathbb{R}^n$ such that $C_{k+1}\subset C_k$, $\cap_{k=0}^{\infty} C_k=C$. Let $\{x^k\}$ be a sequences generated by Algorithm 1. If x^* is a solution of EP(f, C) satisfying x^* is a solution of problems $EP(f,C_k)$ for all $k=0,1,...$ then*

$$\|x^{k+1}-x^*\|^2\le \frac{1}{1-2\lambda_k c_2}\|x^k-x^*\|^2 - \frac{1-2\lambda_k c_1}{1-2\lambda_k c_2}\|x^{k+1}-x^k\|^2,$$

where $\lambda_k<min\{\frac{1}{2c_1},\frac{1}{2c_2}\}$ *for all* $k=0,1,....$

Proof. Since $f(x,y)$ Lipschitz-type on $\mathbb{R}^n$ with two positive constants c_1,c_2, i.e.,

$$f(x,y)+f(y,z)\ge f(x,z)-c_1\|x-y\|^2 -c_2\|y-z\|^2, \quad \forall x,y,z\in C, \quad (2)$$

It is equivalent to

$$\lambda_k\left[f(x^k,x^*)-f(x^k,x^{k+1})\right]\le \lambda_k[f(x^{k+1},x^*) +c_1\|x^k-x^{k+1}\|^2+c_2\|x^{k+1}-x^*\|^2]. \quad (3)$$

Since $x^*\in Sol(f,C_{k+1})$ for all $k=0,1,...$ and $f(x,y)$ is pseudomonotone on $\mathbb{R}^n$ we have $f(x^k,x^*)\le 0, f(x^{k+1},x^*)\le 0.$ (4)

From (2), (3) and (4), it follows that

$$\langle x^{k+1}-x^k, x^{k+1}-x^*\rangle \le \lambda_k[c_1\|x^k-x^{k+1}\|^2 + c_2\|x^{k+1}-x^*\|^2]. \quad (5)$$

On the other hand,

$$\|x^{k+1}-x^*\|^2=\|x^k-x^*\|^2-\|x^{k+1}-x^k\|^2 + 2\langle x^{k+1}-x^k, x^{k+1}-x^*\rangle. \quad (6)$$

This together with (5) implies that

$$(1-2\lambda_k c_2)\|x^{k+1}-x^*\|^2\le\|x^k-x^*\|^2 -(1-2\lambda_k c_1)\|x^{k+1}-x^k\|^2.$$

So, we obtain

$$\|x^{k+1}-x^*\|^2\le \frac{1}{1-2\lambda_k c_2}\|x^k-x^*\|^2 - \frac{1-2\lambda_k c_1}{1-2\lambda_k c_2}\|x^{k+1}-x^k\|^2. \ \square$$

Theorem 3.1.2. *Let* $\{x^k\}$ *be a sequences generated by Algorithm 1,* $0 < \lambda_k < \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\}$ *and* $\sum_{k=1}^{+\infty} \lambda_k < +\infty$. *If there exists* $x^* \in C$ *such that* x^* *is a solution of problems* $EP(f, C_k)$ *for all* $k = 0,1,\ldots$. *Then* $\{x^k\}$ *is weakly convergent to* x^*.

Proof. By Lemma 3.1.1, we have

$$\| x^{k+1} - x^* \|^2 \le \frac{1}{1-2\lambda_k c_2} \| x^k - x^* \|^2,$$

or,

$$\| x^{k+1} - x^* \|^2 \le (1+\alpha_k) \| x^k - x^* \|^2,$$

where $\alpha_k = \frac{2\lambda_k c_2}{1-2\lambda_k c_2}$. Since $\sum_{k=1}^{+\infty} \lambda_k < +\infty$, we have

$$\sum_{k=1}^{+\infty} \alpha_k = \sum_{k=1}^{+\infty} \frac{2\lambda_k c_2}{1-2\lambda_k c_2} < +\infty.$$

By Lemma 2.1, the sequence $\| x^k - x^* \|$ is convergence, and so $\{x^k\}$ is bounded. Thus, there exists a subsequence $\{x^{k_j}\}$ converging to $\bar{x}$. From (1) and (3), it follows that

$$\left\langle x^{k_j+1} - x^k, x^{k_j+1} - x^* \right\rangle \le \lambda_{k_j} [f(x^{k_j+1}, x^*) + c_1 \| x_j^k - x^{k_j+1} \|^2 + c_2 \| x^{k_j+1} - x^* \|^2] + f(x^{k_j}, x^*). \quad (7)$$

From (6) and (7), it follows that

$$(1-2\lambda_{k_j} c_2) \| x^{k_j+1} - x^* \|^2 + (1-2\lambda_{k_j} c_1) \| x^{k_j+1} - x_j^k \|^2 - \| x^{k_j} - x^* \|^2 \le 2\lambda_{k_j} f(x^{k_j+1}, x^*) + f(x^{k_j}, x^*).$$

Using $f(.,y)$ is continuous, let $j \to \infty$, we obtain $f(\bar{x}, x^*) \ge 0$. By the monotonicity of f and $x^* \in \mathrm{Sol}(f, C_k)$, we conclude that

$$0 \le f(x^*, \bar{x}) \le 0.$$

It follows that $x^* = \bar{x}$.

3.2 Algorithm 2

Initialization: Choose the sequence C_k such that $C_k \supseteq C$, $\cap_{k=0}^{\infty} C_k = C$, $C_{k+1} \subset C_k$, $x^0 \in \mathrm{int} C$, positive sequences $\{\lambda_k\}$.

Iteration k:

$$\begin{cases} y^k = \operatorname*{argmin}_{y \in C_k} \left\{ \lambda_k f(x^k, y) + \frac{1}{2} \| y - x^k \|^2 \right\} \\ x^{k+1} = \operatorname*{argmin}_{y \in C_k} \left\{ \lambda_k f(y^k, y) + \frac{1}{2} \| y - x^k \|^2 \right\} \end{cases}$$

Lemma 3.2.1. *We fix sequences* $\{C_k\}$ *of closed convex subsets of* $\mathbb{R}^n$ *such that* $C \subset C_k$, *and* $C_{k+1} \subset C_k \ \forall\, k = 0,1,2\ldots, \cap_{k=0}^{\infty} C_k = C$. *Let* $\{x^k\}$ *be a sequences generated by Algorithm (2). If* x^* *is an solution of (EP) satisfying* x^* *is a solution of problems* $EP(f, C_k)$ *for all* $k = 0,1,\ldots$ *then*

$$\| x^{k+1} - x^* \|^2 \le \| x^k - x^* \|^2 - (1-2\lambda_k c_2) \| x^{k+1} - y^k \| - (1-2\lambda_k c_1) \| x^k - y^k \|^2,$$

for all $k = 0,1,\ldots$.

Proof. Since

$$x^{k+1} = \operatorname*{argmin}_{x \in C_k} \{\lambda_k f(y^k, y) + \frac{1}{2} \| y - x^k \|^2\},$$

we have

$$0 \in \partial_2 \left(\lambda_k f(y^k, y) + \frac{1}{2} \| y - x^k \|^2 \right)(x^{k+1}) + N_{C_k}(x^{k+1}).$$

Thus, there exist $w \in \partial_2 f(y^k, x^{k+1})$ and $\bar{w} \in N_C(x^{k+1})$ such that

$$\lambda_k w + x^{k+1} - x^k + \bar{w} = 0.$$

Hence, it follows from the definition of N_{C_k} that

$$\left\langle \lambda_k w + x^{k+1} - x^k, x - x^{k+1} \right\rangle = \lambda_k \left\langle w, x - x^{k+1} \right\rangle - \left\langle x^{k+1} - x^k, x^{k+1} - x \right\rangle \ge 0, \ \forall x \in C_k.$$

So,

$$\left\langle x^{k+1} - x^k, x^{k+1} - x \right\rangle \le \lambda_k \left\langle w, x - x^{k+1} \right\rangle \quad \forall x \in C_k. \quad (8)$$

By $w \in \partial_2 f(y^k, x^{k+1})$, we have

$$\left\langle w, x - x^{k+1} \right\rangle \le f(y^k, x) - f(y^k, x^{k+1}), \quad \forall x \in \mathbb{R}^n. \quad (9)$$

From relations (8) and (9), we obtain

$$\lambda_k \left(f(y^k, x) - f(y^k, x^{k+1}) \right) \ge \left\langle x^{k+1} - x^k, x^{k+1} - x \right\rangle, \forall x \in C_k. \quad (10)$$

Since $x^* \in C_k$, we have

$$\left\langle x^{k+1} - x^k, x^{k+1} - x^* \right\rangle \le \lambda_k \left[f(y^k, x^*) - f(y^k, x^{k+1}) \right].$$

Since $x^* \in Sol(f, C_k)$ and $f(x,y)$ is pseudomonotone we have $f(y^k, x^*) \le 0$.

This together with (10) implies that

$$\left\langle x^{k+1}-x^k, x^*-x^{k+1}\right\rangle \geq \lambda_k f(y^k, x^{k+1}) \qquad (11)$$

Since $f(x,y)$ Lipschitz-type on R^n with two positive constants c_1, c_2, i.e.,

$$f(x,y)+f(y,z)\geq f(x,z)- c_1\|x-y\|^2 - c_2\|y-z\|^2, \forall x,y,z.$$

Appying above inequality with $x=x^k$ $y=y^k$, $z=x^{k+1}$ we have

$$f(y^k,x^{k+1})\geq f(x^k,x^{k+1})-f(x^k,y^k) - c_1\|y^k-x^k\|^2 - c_2\|x^{k+1}-y^k\|^2.$$

This together with (11) implies that

$$\left\langle x^{k+1}-x^k, x^*-x^{k+1}\right\rangle \geq \lambda_k[f(x^k,x^{k+1})- f(x^k,y^k)-c_1\|y^k-x^k\|^2 - c_2\|x^{k+1}-y^k\|^2]. \qquad (12)$$

Since

$$y^k = \underset{y\in C_k}{\operatorname{argmin}}\left\{\lambda_k f(x^k,y)+\frac{1}{2}\| y-x^k\|^2\right\},$$

by an argument similar to the above, we obtain

$$\left\langle y^k-x^k, y^k-y\right\rangle \geq \lambda_k\left[f(x^k,y)-f(x^k,y^k)\right],$$

$\forall y\in C_k$. Appying above inequality with $y=x^{k+1}$, we get

$$\left\langle y^k-x^k, y^k-x^{k+1}\right\rangle \geq \lambda_k\left[f(x^k,x^{k+1})-f(x^k,y^k)\right]$$

This together with (12) and

$$\|x^{k+1}-x^*\|^2=\|x^k-x^*\|^2-\|x^{k+1}-x^k\|^2 +2\left\langle x^{k+1}-x^k, x^{k+1}-x^*\right\rangle,$$

implies that

$$\|x^k-x^*\|^2-\|x^{k+1}-x^k\|^2-\|x^{k+1}-x^*\|^2\geq 2\left\langle y^k-x^k, y^k-y\right\rangle - 2c_1\lambda_k\|y^k-x^k\|^2 - 2c_2\lambda_k\|x^{k+1}-y^k\|^2.$$

So that,

$$\begin{aligned}\|x^{k+1}-x^*\|^2 &\leq \|x^k-x^*\|^2-\|x^{k+1}-x^k\|^2 -2\left\langle y^k-x^k, y^k-x^{k+1}\right\rangle + 2c_1\lambda_k\|y^k-x^k\|^2 + 2c_2\lambda_k\|x^{k+1}-y^k\|^2\\ &= \|x^k-x^*\|^2-\|(x^{k+1}-y^k)-(x^k-y^k)\|^2 -2\left\langle y^k-x^k, y^k-x^{k+1}\right\rangle + 2c_1\lambda_k\|y^k-x^k\|^2 + 2c_2\lambda_k\|x^{k+1}-y^k\|^2\\ &= \|x^k-x^*\|^2-(1-2\lambda_k c_1)\|y^k-x^k\|^2 -(1-2\lambda_k c_2)\|x^{k+1}-y^k\|^2.\end{aligned}$$

Theorem 3.2.2. Assume that (A_1)-(A_4) hold, sequence $\{\lambda_k\}$ satisfies $\lim_{k\to+\infty}\lambda_k=\lambda$ $0<\rho<\lambda_k\leq\min\{\frac{1}{2c_1},\frac{1}{2(c_2-\gamma)}\}$, and there exists $x^*\in C$ such that x^* is a solution of problems $EP(f,C_k)$ for all $k=0,1,\ldots$. Then $\left\{x^k\right\}$ is weakly convergent to x^*.

Proof. Using the assumption of $\{\lambda_k\}$, we have $1-2\lambda_k c_1>0$ and $1-2\lambda_k c_2>0$. By using Lemma 3.2.1, we obtain

$$\|x^{k+1}-x^*\|\leq\|x^k-x^*\|, \forall k\geq 0.$$

It follows that $\{\|x^k-x^*\|\}$ is nondecreasing and lower bounded by 0, and so $\lim_{k\to+\infty}\|x^k-x^*\|=c<+\infty$. Hence, the $\{\|x^k-x^*\|\}$ is bounded and there exists a subsequence $\{x^{k_j}\}$ converging to $\bar{x}$. Since $C_{k+1}\subset C_k$ $\forall k=0,1,2...$ and $\cap_{k=0}^{\infty}C_k=C$, it is easily to check that $\bar{x}\in C$. By using Lemma 3.2.1, we get

$$(1-2\rho c_1)\|y^k-x^k\|^2\leq(1-2\lambda_k c_1)\|y^k-x^k\|^2 \leq\|x^k-x^*\|^2-\|x^{k+1}-x^*\|^2.$$

Due to $1-2\rho c_1>0$ and $\lim_{k\to+\infty}\|x^k-x^*\|=c$, it follows that $\lim_{k\to+\infty}\|x^k-y^k\|=0$. This implies that $\{y^{k_j}\}$ convergent to $\bar{x}$ because $\{x^{k_j}\}$ convergent to $\bar{x}$. On the other hand

$$y^{k_j}=\underset{y\in C_{k_j}}{\operatorname{argmin}}\left\{\lambda_{k_j} f(x^{k_j},y)+\frac{1}{2}\| y-x^{k_j}\|^2\right\}.$$

Since f is lower semicontinuous on $C\times C$, letting $j\to+\infty$ yields

$$\bar{x}=\operatorname{argmin}\left\{\lambda f(\bar{x},y)+\frac{1}{2}\|y-\bar{x}\|^2 : y\in C_{k_j}\right\}.$$

It follows that $\bar{x}$ is a solution of problems $EP(f,C_{k_j})$ for all $j=0,1,...$ and a solution of the problem (EP) (because $C\subset C_{k_j}$). From $\lim_{k\to+\infty}\|x^{k_j}-x^*\|=c$ and replacing x^* by $\bar{x}$ we have $c=\lim_{k\to+\infty}\|x^{k_j}-\bar{x}\|=0$.

Hence, the $\{x^k\}$ convergent to $\bar{x}$.

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CONCLUSIONS

we present two new outer approximation algorithms for solving equilibrium problems on a convex subset C with the cost function $f(x, y)$ is pseumonotone and satisfies Lipschitz-type condition on $\mathbb{R}^n$ with two constants c_1 and c_2. The algorithm is to define proximal point subproblems or projector point subproblems on the convex domains $C_k \supseteq C$, $C_{k+1} \subset C_k, \cap_{k=0}^{\infty} C_k = C,\ k = 0,1,...$ Under standard assumptions imposed on cost mappings we obtained strongly convergence theorem of the algorithms.

THUẬT TOÁN XẤP XỈ NGOÀI GIẢI BÀI TOÁN CÂN BẰNG

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Trường Đại học Điện lực

Tóm tắt: *Gần đây, các bài toán cân bằng được ứng dụng một cách rộng rãi và hiệu quả trong nhiều lĩnh vực thực tế, như vật lý, kỹ thuật, lý thuyết trò chơi, giao thông, kinh tế và mạng internet. Bài toán đã thu hút nhiều nhà khoa học tham gia nghiên cứu cả về lý thuyết lẫn ứng dụng. Bài báo này quan tâm nghiên cứu phương pháp tìm nghiệm xấp xỉ của bài toán cân bằng. Sử dụng phương pháp phát triển lý thuyết, chúng tôi đưa ra hai thuật toán xấp xỉ ngoài tìm nghiệm gần đúng của bài toán cân bằng trên tập con lồi C với gải thiết hàm giá là liên tục kiểu Lipschitz và giả đơn điệu. Thuật toán được xác định bởi các dãy các bài toán điểm gần kề hay bài toán tìm điểm chiếu trên các miền lồi $C_k \supseteq C,\ \ C_{k+1} \subset C_k, \cap_{k=0}^{\infty} C_k = C,\ k = 0,1,...$ Với các giả thiết cơ bản của hàm giá, chúng tôi đã thu các định lý hội tụ mạnh của các thuật toán trong bài báo.*

Từ khóa: *Bài toán cân bằng; Bài toán điểm gần kề; Giả đơn điệu; Liên tục kiểu Lipschitz; Thuật toán xấp xỉ ngoài.*

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