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A CHARACTERIZATION OF A CLASS OF *M*-SUBHARMONIC FUNCTIONS AND APPLICATIONS

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Abstract. *In this paper we give the properties for m-capacity which is a generalization of the ones in [13]. Then as an application, we introduce a characterization of* $\mathcal{E}_m^0$ *class in terms of m-capacity.*

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1. INTRODUCTION

In 1985, Caffarelli, Nirenberg and Spruck [10] proposed a model that made it possible to study the common properties of potential and pluripotential theories, as well as the transition between them. The core focuses on the this framework is what is known today as the class of *m*-subharmonic functions, which are extensions of the class of plurisubharmonic functions and encompasses the subharmonic and plurisubharmonic ones naturally. Recently, many authors have studied the class of *m*-subharmonic functions (see [9], [5], [12], [14]).

In order to study the complex *m*-Hessian operator for *m*-subharmonic functions which were not locally bounded on *m*-hyperconvex domain, Chinh introduced the Cegrell classes (in [5]); moreover, he proved that the complex *m*-Hessian operator was well-defined in these classes.

In the class of plurisubharmonic functions, the weighted energy class $\mathcal{E}_\chi(\Omega)$ that was a generalization of the class $\mathcal{E}_p(\Omega)$ has been introduced and studied by Benelkourchi, Guedj and Zeriahi in [2] and [8]. In the class of *m*-subharmonic functions, we has introduced the class $\mathcal{F}_{m,\chi}(\Omega)$ with finite weighted complex *m*-Hessian [6].

In this paper, we shall construct a new class of weighted energy functions in the class of *m*-subharmonic functions and study some analysis properties of this class. Specifically, the paper is organized as follows. The paper has three sections. In Section 2, we give some definitions and necessary results in order to prove our main results. Our main results are stated and proved in Section 3. We shall prove a characterization in finite weighted energy in term of C_m-capacity for the class $\mathcal{E}_m^0(\Omega)$.

2. Preliminaries

In this section, we review some of definitions and results concerning to *m*-subharmonic functions which have been introduced and investigated intensively in recent years by many authors (see [9], [12], [14]). We also recall the Cegrell classes of *m*-subharmonic functions $\mathcal{F}_m(\Omega)$ and $\mathcal{E}_m(\Omega)$ introduced and studied in [5]. Next, we recall the class $\mathcal{F}_{m,\chi}(\Omega)$ with finite weighted complex *m*-Hessian was introduced in [6].

2.1 The class of *m*-subharmonic functions

First, we will denote by $\mathbb{C}_{(1,1)}$ the space of *(1,1)*-forms with constant coefficients.

For $1\le m\le n$, we define

$$\hat{\Gamma}_m=\{\eta\in\mathbb{C}_{(1,1)}:\eta\wedge\beta^{n-1}\ge 0,\dots,\eta^m\wedge\beta^{n-m}\ge 0\}.$$

Definition 2.1. Let u be a subharmonic function on an open subset $\Omega\subset\mathbb{C}^n$. Then u is said to be a *m*-subharmonic function on Ω iff for every $\eta_1,\dots,\eta_{m-1}$ in $\hat{\Gamma}_m$ the inequality

$$dd^cu\wedge\eta_1\wedge\cdots\wedge\eta_{m-1}\wedge\beta^{n-m}\ge 0,$$

holds in the sense of currents. We also denote $SH_m(\Omega)$ the set of *m*-subharmonic functions on Ω while $SH_m^-(\Omega)$ denotes the set of negative *m*-subharmonic functions on Ω.

Due to Proposition 3.1 in [9] (also see the Definition 1.2 in [14]), a subharmonic function u is *m*-subharmonic on Ω iff $(dd^cu)^k\wedge\beta^{n-k}\ge 0,$ for $k=1,\dots,m$. More generally, if $u_1,\dots,u_k\in C^2(\Omega)$, then for all $\eta_1,\dots,\eta_{m-k}\in\hat{\Gamma}_m$, we have

$$dd^cu_1\wedge\cdots\wedge dd^cu_k\wedge\eta_1\wedge\cdots\wedge\eta_{m-k}\wedge\beta^{n-m}\geq 0 \quad (1)$$

holds in the sense of currents.

It follows on [9] and [12] the m-Hessian operator for locally bounded m-subharmonic functions is defined as follows.

Definition 2.2. Assume that $u_1,\ldots,u_p\in \mathrm{SH}_m(\Omega)\cap L^\infty_{\mathrm{loc}}(\Omega)$ with $p\leq m$. Then the *complex Hessian operator* $H_m(u_1,\ldots,u_p)$ is defined inductively by

$$dd^cu_p\wedge\cdots\wedge dd^cu_1\wedge\beta^{n-m}=dd^c(u_p dd^cu_{p-1}\wedge\cdots\wedge dd^cu_1\wedge\beta^{n-m})\}.$$

In [9] and [12], the authors proved that $H_m(u_1,\ldots,u_p)$ is a closed positive current of bidegree $(n-m+p,n-m+p)$ and this operator is continuous under decreasing sequences of locally bounded m-subharmonic functions. In particular, when $u=u_1=\cdots=u_m\in \mathrm{SH}_m(\Omega)\cap L^\infty_{\mathrm{loc}}(\Omega)$ the Borel measure $H_m(u)=(dd^cu)^m\wedge\beta^{n-m}$ is well defined and is called the complex m-Hessian of u.

2.2 Some weighted energy classes of *m*-subharmonic

Now, we recall the following weighted energy classes of m-subharmonic functions which were introduced and investigated in [5]. From now on, we make the assumption that Ω is a bounded m-hyperconvex domain in $\mathbb{C}^n$, there exists a continuous m-subharmonic function $u:\Omega\to\mathbb{R}^-$ such that $\Omega_c=\{u<c\}\Subset\Omega,\ \forall c<0$. It has already been shown that, every plurisubharmonic function is m-subharmonic with $m\geq 1$, so every m-hyperconvex domain in $\mathbb{C}^n$ is hyperconvex. We put

$$\mathrm{E}^0_m(\Omega)=\{u\in SH^-_m(\Omega)\cap L^\infty(\Omega):\lim_{z\to\partial\Omega}u(z)=0,\int_\Omega H_m(u)<+\infty\},$$

$$\mathrm{F}_m(\Omega)=\{u\in SH^-_m(\Omega):\exists u_j\in\mathrm{E}^0_m,u_j\searrow u,\sup\int_\Omega H_m(u_j)<+\infty\}$$

and

$$\mathrm{E}_m(\Omega)=\Big\{u\in SH^-_m(\Omega):\forall z_0\in\Omega,\text{and }\exists\text{ a neighborhood }\ \omega\ni z_0,\ \text{ and } \mathrm{E}^0_m\ni u_j\searrow u\text{ on }\omega,\sup_j\int_\Omega H_m(u_j)<\infty\Big\}.$$

From Theorem 3.14 in [5] it follows that if $u\in\mathrm{E}_m(\Omega)$, the complex m-Hessian $H_m(u)=(dd^cu)^m\wedge\beta^{n-m}$ is well defined and it is a Radon measure on Ω. On the other hand, by Remark 3.6 in [5] the following description of $\mathrm{E}_m(\Omega)$ may be given

$$\mathrm{E}_m=\mathrm{E}_m(\Omega)=\Big\{u\in SH^-_m(\Omega):\forall U\Subset\Omega,\exists v\in\mathrm{F}_m(\Omega),v=u\text{ on }U\Big\}. \quad (2)$$

Now we recall the two weighted pluricomplex energy classes of plurisubharmonic functions defined in [6].

Definition 2.3. Let $\chi:\mathbb{R}^-\to\mathbb{R}^+$ be an decreasing function and $1\leq m\leq n$ and Ω is a bounded m-hyperconvex domain in $\mathbb{C}^n$. We define

$$\mathrm{F}_{m,\chi}(\Omega)=\{u\in SH^-_m(\Omega):\exists\ \{u_j\}\subset\mathrm{E}^0_m,\ u_j\searrow u \text{ on } \Omega$$
$$\sup_j e^m_\chi(u_j)=\sup_j\int_\Omega\chi(u_j)(dd^cu_j)^m\wedge\beta^{n-m}<+\infty\},$$

and

$$\mathrm{E}_{m,\chi}(\Omega)=\{u\in SH^-_m(\Omega):\forall\ K\Subset\Omega,\exists v\in\mathrm{F}_{m,\chi},v=u \text{ on } K\}.$$

In the case $\chi(t)\equiv 1$ for all *t < 0*, we get $\mathrm{F}_{m,\chi}(\Omega)(resp.\mathrm{E}_{m,\chi}(\Omega))$ coincide with the class $\mathrm{F}(\Omega)(resp.\mathrm{E}(\Omega))$ in [4]. Moreover, by Proposition 4 in [6], $\mathrm{E}_{m,\chi}(\Omega)\subset\mathrm{E}_m(\Omega)$ provided that $\chi(2t)\leq a\chi(t)$ for all *t < 0* with some *a > 1*. We will prove the this assertion in another way as follows.

Proposition 2.4. If $\chi(2t)\leq a\chi(t)$ for all *t < 0* with some *a > 1* then

$$\mathrm{F}_{m,\chi}(\Omega)\subset\mathrm{F}_m(\Omega)\ \ (resp.\mathrm{E}_{m,\chi}(\Omega)\subset\mathrm{E}_m(\Omega)).$$

Proof. We prove in the second case, that $\varphi\in\mathrm{E}_{m,\chi}(\Omega)$ then φ in $\mathrm{E}_m(\Omega)$. Indeed, we can assume without loss of generality that $\chi(0)=0$ and $\chi(t)>0$ for every *t < 0*. Let $z_0\in\Omega$ and take a neighbourhood $K\Subset\Omega$ of z_0 and a sequence $\{\varphi_j\}\subset\mathrm{E}^m_0(\Omega)$ such that $\varphi_j\searrow\varphi$ on *K* and

$$\sup_j \int_\Omega \chi(\varphi_j) H_m(\varphi_j) < \infty.$$

Choose a convex decreasing function χ such that $\chi \le \chi$. Then

$$dd^c \chi(\varphi) = \chi''(\varphi) d\varphi \wedge d^c\varphi \wedge T + \chi'(\varphi) dd^c\varphi \wedge T \ge 0, \quad (3)$$

where T is closed positive current of bidegree *(n -1, n -1)*, hence $\chi(\varphi) \in SH_m^-(\Omega)$. Then

$$\sup_{j\in\mathbb{N}} \int_\Omega \chi(\varphi) H_m(\varphi_j) \le \sup_{j\in\mathbb{N}} \int_\Omega \chi(\varphi_j) H_m(\varphi_j) \le \sup_{j\in\mathbb{N}} \int_\Omega \chi(\varphi_j) H_m(\varphi_j) < +\infty$$

Now, as in Lemma 5.5 in [13], we have $\varphi_j^K = \sup\{u \in SH_m^-(\Omega) : u \le (II_K)\varphi_j\} \in \mathcal{E}_m^0(\Omega)$.

$$\int_\Omega -h_0 H_m(\varphi_j^K) \le \int_\Omega -h_0 H_m(\varphi_j).$$

It follows from (3), $\chi(\varphi) \in SH_m^-(\Omega)$, we have

and $\varphi_j^G \nearrow \varphi$ on K. Let $h_0 \in \mathcal{E}_m^0(\Omega)$ with $h_0 \not\equiv 0$ such that $-h_0 \le \chi(\varphi)$. Then by Lemma 2.6 in [13], we have

$$\int_\Omega -h_0 H_m(\varphi_j) \le \sup_{j\in\mathbb{N}} \int_\Omega \chi(\varphi) H_m(\varphi_j) < +\infty$$

Now

$$\inf_K (-h_0) \sup_{j\in\mathbb{N}} \int_K H_m(\varphi_j^K) \le \sup_{j\in\mathbb{N}} \int_K -h_0 H_m(\varphi_j^K) \le \sup_{j\in\mathbb{N}} \int_\Omega -h_0 H_m(\varphi_j^K)$$

Hence

$$\sup_{j\in\mathbb{N}} \int_\Omega H_m(\varphi_j^K) \le \frac{1}{-\sup_K h_0} \sup_{j\in\mathbb{N}} \int_\Omega -h_0 H_m(\varphi_j^K) < +\infty,$$

Thus $\lim_j \varphi_j^K \in \mathcal{F}_m(\Omega)$. By (2), we have that $\varphi \in \mathcal{E}_m(\Omega)$.

2.3. The *m*-capacity

We recall the notion of *m*-capacity as in [5]. As well as C_n capacity in the sense of Bedford and Taylor on Ω, the C_m -capacity is the set function which is given as follows.

Definition 2.5. The *m*-capacity of a Borel set $E \subset \Omega$ with respect to Ω, is defined by

$$C_m(E) = C_m(E,\Omega) = \sup\left\{\int_E H_m(u) : u \in SH_m(\Omega), -1 \le u \le 0\right\}.$$

The outer *m*-capacity of a Borel set $E \subset \Omega$ is defined by

$$C_m^*(E,G) := \inf\{C_m(G,\Omega) : E \subset G, G$$

is an open subset of Ω }.

We refer to [6] for more details about elementary properties of the *m*-capacity, which is similar to the capacity was presented in [1] (see also [3]).

On other hand, let $h_{E,\Omega}^*$ denotes the smallest upper semicontinuous majorant of $h_{E,\Omega}$. It follows on Theorem 2.20 in [11], we get

$$C_m^*(E) = \int_\Omega H_m(h_{E,\Omega}^*)$$

and

$$h_{E,\Omega}(z) = \sup\{\varphi(z) : \varphi \in SH_m^-(\Omega), \varphi \le -1 \text{ on } E\}.$$

Now we combine with Proposition 2.10 in [5], we gives some well-known and elementary properties of the *m*-capacity similar to the capacity presented in [1] (see also [3]). Namely, we have the following resutl, which is a generalization of the ones in [13].

Lemma 2.6.

a) If $E_1 \subset E_2 \subset \Omega_1 \subset \Omega_2$ then $C_m(E_1,\Omega_2) \le C_m(E_2,\Omega_1)$;

b) $C_m(\bigcup_{j=1}^\infty E_j) \le \sum_{j=1}^\infty C_m(E_j)$;

c) If $E \subset D \Subset \Omega_1 \subset \Omega_2 \Subset \mathbb{C}^n$ then $C_m(E,\Omega_1) \le C_{D,\Omega_1,\Omega_2} C_m(E,\Omega_2)$;

d) If $E_j \nearrow E$ then $C_m(E_j) \nearrow C_m(E)$.

Proof. The assertion a), b) and d) are follows from the definition of the *m*-capacity. Now, we prove c). Indeed, since we can cover D by finite number of balls contained in Ω_1 then by a) and b) we can assume that $D = B(z_0, r)$ and $\Omega_1 = B(z_0, R)$, where $B(z,r)$ is the ball with center at z and radius

r. Now, let $u \in SH_m(\Omega_1)$ be such that $-1 \le u \le 0$. We set

$$\psi(z) = (R^2 - r^2)^{-1}(|z - z_0|^2 - R^2).$$

Then $\psi \in SH_m(\mathbb{C}^n), \psi = 0$ on $\partial\Omega_1$, $\psi \le -1$ on D. Set

$$u = \begin{cases} \max\{u, \psi\} & \text{on } \Omega_1, \\ \psi & \text{on } \Omega_2 \setminus \Omega_1 \end{cases}$$

and $v = (1+c)^{-1}(u - c)$, $c = \|\psi\|_{L^\infty(\Omega_2)}$. Then we have

$$v \in SH_m(\Omega_2), -1 \le v \le 0 \text{ and } H_m(v) = (1+c)^{-m} H_m(u) \text{ on } D.$$

Hence $C_m(E, \Omega_1) \le (1+c)^m C_m(E_2, \Omega_2)$.

From Theorem 3.6 in [13], we have we give the following.

Proposition 2.7. If $u, v \in \mathcal{E}_m^0(\Omega)$ then

$$\int_{\{u>v\}} (dd^c u)^m \wedge \beta^{n-m} = \int_{\{u>v\}} (dd^c \max(u,v))^m \wedge \beta^{n-m}.$$

Proof. First we prove the following result

$$\int_\Omega (dd^c \max(u,v))^m \wedge \beta^{n-m} \le \int_\Omega (dd^c u)^m \wedge \beta^{n-m} \qquad (4)$$

Indeed, for all $h \in C_o^\infty(\Omega)$, put $\omega = \max\{u, v\} \in \mathcal{F}_m(\Omega)$, then by Stokes formula we have

$$\begin{aligned}\int_\Omega -h(dd^c\omega)^m \wedge \beta^{n-m} &= \int_\Omega -\omega dd^c h \wedge (dd^c\omega)^{m-1} \wedge \beta^{n-m} \\ &\le \int_\Omega -u dd^c h \wedge (dd^c\omega)^{m-1} \wedge \beta^{n-m} \\ &= \int_\Omega -h dd^c u \wedge (dd^c\omega)^{m-1} \wedge \beta^{n-m} \\ &= \cdots = \int_\Omega -h(dd^c u)^{m-1} \wedge \beta^{n-m}.\end{aligned}$$

Then let $h \to -1$ we have the desired.

Now, applying Theorem 3.6 in [13], we have

$$(dd^c u)^m \wedge \beta^{n-m} = (dd^c \max(u,v))^m \wedge \beta^{n-m}$$

on $\{u>v\}$, we imply that

$$\begin{aligned}\int_\Omega (dd^c \max(u,v))^m \wedge \beta^{n-m} &= \int_{\{u>v\}} (dd^c \max(u,v))^m \wedge \beta^{n-m} + \int_{\{u \le v\}} (dd^c \max(u,v))^m \wedge \beta^{n-m} \\ &\le \int_{\{u>v\}} (dd^c u)^m \wedge \beta^{n-m} + \int_{\{u \le v\}} (dd^c \max(u,v))^m \wedge \beta^{n-m}\end{aligned}$$

Then, from (4), we have

$$\int_{\{u<v\}} (dd^c v)^m \wedge \beta^{n-m} \le \int_{\{u \le v\}} (dd^c u)^m \wedge \beta^{n-m} \qquad (5)$$

Then, replace v by $v - \varepsilon$ in (5) we obtain the following

$$\int_{\{u<v-\varepsilon\}} (dd^c v)^m \wedge \beta^{n-m} \le \int_{\{u \le v-\varepsilon\}} (dd^c u)^m \wedge \beta^{n-m} \le \int_{\{u<v\}} (dd^c u)^m \wedge \beta^{n-m}$$

Therefore, let $\varepsilon \to 0$, we have the following

$$\int_{\{u<v\}} (dd^c v)^m \wedge \beta^{n-m} \le \int_{\{u<v\}} (dd^c u)^m \wedge \beta^{n-m}.$$

and the desired conclusion follows. The proof of the theorem is complete.

3. Results

In this section, we give the following properties for the weighted m-capacity of the functions in $\mathcal{E}_m^0(\Omega)$.

Proposition 2.8. Forall $u\in E_m^0(\Omega)$ and $s\geq 0$ we have

$$s^m C_m(\varphi<-2s)\leq \int_{\{\varphi<-s\}}(dd^c\varphi)^m\wedge\beta^{n-m}\leq s^m C_m(\varphi<-s).$$

Proof. Next, by Lemma 2.6 and then applying Theorem 3.6 in [13], we have

$$1_{\{u>v\}}(dd^c u)^m\wedge\beta^{n-m}=1_{\{u>v\}}(dd^c\max(u,-t))^m\wedge\beta^{n-m}.$$

Then

$$\begin{aligned}\int_{\{u<-s\}}(dd^c u)^m\wedge\beta^{n-m}&=\int_{\{u<-s\}}(dd^c\max(u,-s))^m\wedge\beta^{n-m}\\&=s^m\int_{\{u<-s\}}(dd^c\max(\frac{u}{s},-1))^m\wedge\beta^{n-m}\\&\leq s^m C_m(u<-s)\end{aligned}$$

this proof the first inequality.

To complete the proof it remains to show the second inequality. We have

$$\begin{aligned}C_m(u<-2t)&=\sup\{\int_{\{u<-2t\}}(dd^c\varphi)^m\wedge\beta^{n-m}:\varphi\in SH_m(\Omega),-1\leq\varphi\leq 0\}\\&=\sup\{\int_{\{\frac{u}{t}<-2\}}(dd^c\varphi)^m\wedge\beta^{n-m}:\varphi\in SH_m(\Omega),-1\leq\varphi\leq 0\}\\&\leq\sup\{\int_{\{\frac{u}{t}<\varphi-1\}}(dd^c\varphi)^m\wedge\beta^{n-m}:\varphi\in SH_m(\Omega),-1\leq\varphi\leq 0\}\end{aligned}$$

Then, apply Proposition 2.7 we have

$$\int_{\{\frac{u}{t}<\varphi-1\}}(dd^c\varphi)^m\wedge\beta^{n-m}\leq\int_{\{\frac{u}{t}<\varphi-1\}}(dd^c\frac{u}{t})^m\wedge\beta^{n-m}.$$

Then

$$\begin{aligned}C_m(u<-2t)&\leq\sup\{\int_{\{\frac{u}{t}<\varphi-1\}}(dd^c\varphi)^m\wedge\beta^{n-m}:\varphi\in SH_m(\Omega),-1\leq\varphi\leq 0\}\\&\leq\sup\{\int_{\{\frac{u}{t}<\varphi-1\}}(dd^c\frac{u}{t})^m\wedge\beta^{n-m}:\varphi\in SH_m(\Omega),-1\leq\varphi\leq 0\}\\&\leq\frac{1}{t^m}\int_{\{\frac{u}{t}<-1\}}(dd^c\frac{u}{t})^m\wedge\beta^{n-m}=\frac{1}{t^m}\int_{\{u<-t\}}(dd^c u)^m\wedge\beta^{n-m}\end{aligned}$$

Thus

$$t^m C_m(u<-2t)\leq\int_{\{u<-t\}}(dd^c u)^m\wedge\beta^{n-m}$$

the desired conclusion follows. The proof of the theorem is complete.

The E_m^0 and were defined by m-Hessian measure. By using Proposition 2.8 about the upper bound as m-capacity in the class E_m^0, we gave a characterization for this class in th following Theorem, which is the main result of this paper.

Theorem 2.9. Asusume that $\chi:\mathbb{R}^-\to\mathbb{R}^+$ is a function satissfiles the condition $\chi(2t)\leq a\chi(t), a>0$ and $\chi'(t)\leq 0,\forall t<0$. Then $\forall\varphi\in E_m^0$ we have

$$e_\chi^m(\varphi)\sim\int_0^\infty -\chi'(-s)s^m C_m(\varphi<-s)ds.$$

Proof. Indeed, applying Lema 1 in [6], we have

$$s^m C_m(\varphi < -2s) \le \int\limits_{\{\varphi<-s\}} (dd^c\varphi)^m \wedge \beta^{n-m} \le s^m C_m(\varphi < -s).$$

Note that, by replacing $\min(\chi(t); -jt)$ and approximately if necessary for $\chi(t)$, we can assume that $\chi(0) = 0$. Then

$$e_\chi^m(\varphi) = \int\limits_\Omega \chi(\varphi)(dd^c\varphi)^m \wedge \beta^{n-m} = \int\limits_0^\infty -\chi'(-s) \int\limits_{\{\varphi<-s\}} (dd^c\varphi)^m \wedge \beta^{n-m} ds.$$

Then

$$\int\limits_0^\infty -\chi'(-s) s^m C_m(\varphi < -2s) \le e_\chi^m(\varphi) \le \int\limits_0^\infty -\chi'(-s) s^m C_m(\varphi < -s).$$

On other hand

$$\int\limits_0^\infty -\chi'(-s) \int\limits_{\{\varphi<-2s\}} (dd^c\varphi)^m \wedge \beta^{n-m} ds = \int\limits_0^\infty -\chi'(-s) \int\limits_{\{\frac{\varphi}{2}<-s\}} (dd^c\varphi)^m \wedge \beta^{n-m} ds$$

$$= \frac{1}{2^{m+1}} \int\limits_0^\infty -\chi'(-\frac{s}{2}) \int\limits_{\{\varphi<-s\}} (dd^c\varphi)^m \wedge \beta^{n-m} ds$$

However, we have

$$\int\limits_0^\infty \chi(\frac{\varphi}{2})(dd^c\varphi)^m \wedge \beta^{n-m} ds = \frac{1}{2} \int\limits_0^\infty -\chi'(-\frac{s}{2}) \int\limits_{\{\varphi<-s\}} (dd^c\varphi)^m \wedge \beta^{n-m} ds.$$

Hence, applying the condition $\chi(2t) \le a\chi(t), a > 0$, we have

$$\int\limits_0^\infty -\chi'(-s) \int\limits_{\{\varphi<-2s\}} (dd^c\varphi)^m \wedge \beta^{n-m} ds = \frac{1}{2^{m+1}} \int\limits_0^\infty \chi(\frac{\varphi}{2})(dd^c\varphi)^m \wedge \beta^{n-m} ds$$

$$\ge \frac{1}{2^m} \int\limits_0^\infty \chi(\frac{\varphi}{2})(dd^c\varphi)^m \wedge \beta^{n-m} ds$$

$$\ge \frac{1}{a2^m} \int\limits_0^\infty \chi(\varphi)(dd^c\varphi)^m \wedge \beta^{n-m} ds$$

From which we infer that

$$\int\limits_0^\infty -\chi'(-s) s^m C_m(\varphi < -2s) \ge \frac{1}{a2^m} \int\limits_0^\infty -\chi'(-s) s^m C_m(\varphi < -s).$$

Thus

$$e_\chi^m(\varphi) \sim \int\limits_0^\infty -\chi'(-s) s^m C_m(\varphi < -s) ds.$$

The proof of the theorem is complete.

Corollary 2.10. Assume that χ satisfies the condition of Theorem 2.9 and $v \in SH_m(\Omega), u \in \mathrm{F}_{m,\chi}(\Omega), u \le v$ then $v \in \mathrm{F}_{m,\chi}(\Omega)$.

Proof. Since $u \in \mathrm{F}_{m,\chi}$, then from Definition 2.3 that

$$\exists \varphi_j \in \mathrm{E}_m^0, \varphi_j \searrow u, \sup_j e_\chi^m(\varphi) < +\infty.$$

Put $v_j = \max(\varphi_j, v)$, then $v_j \in SH_m(\Omega)$. We prove that $v_j \in \mathrm{E}_m^0$. Indeed, since $\varphi_j \searrow u$, but $v \ge u$ we infer that $v_j \searrow v$ and $\lim\limits_{z\to\partial\Omega} v(z) = 0$. Further more, by Theorem 5.2 in [7]

$$\int\limits_\Omega (dd^c v_j)^m \wedge \beta^{n-m} \le \int\limits_\Omega (dd^c \varphi_j)^m \wedge \beta^{n-m} < \infty.$$

To prove the theorem, we only need to prove that

$$\sup_j \int\limits_\Omega \chi(v_j)(dd^c v_j)^m \wedge \beta^{n-m} < \infty.$$

Indeed, we have from Theorem 2.9

$$\int_\Omega \chi(v_j)(dd^c v_j)^m \wedge \beta^{n-m} = \int_0^\infty -\chi'(-s)s^m C_m(v_j < -s)ds \le \int_0^\infty -\chi'(-s)s^m C_m(\varphi_j < -s)ds \sim e_\chi^m(\varphi_j)$$

Then

$$\sup_j e_\chi^m(v_j) \le \sup_j e_\chi^m(\varphi) < +\infty.$$

Thus $v \in \mathcal{F}_{m,\chi}$.

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ĐẶC TRƯNG CHO MỘT LỚP HÀM *m*-ĐIỀU HÒA DƯỚI

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Tóm tắt: *Chúng tôi đưa ra một số đánh giá đối với m-dung tích, từ đó đưa ra một đặc trưng của lớp* $\mathcal{E}_m^0$ *thông qua m-dung tích.*

Từ khóa: *Hàm đa điều hòa dưới, hàm m-điều hòa dưới, miền siêu lồi, toán tử m-Hessian, toán tử Monge-Ampere, hàm trọng, m-dung tích.*

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