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SOME PROPERTIES OF A CLASS OF DELTA m- SUBHARMONIC FUNCTIONS

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Abstract. *In this paper, we introduce a class of weighted energy in the class of m-subharmonic functions* $\delta \mathcal{W}_{m,\chi}(\Omega)$. *We also shown that this class is convex cone, stable under maximum and* $\delta \mathcal{W}_{m,\chi}(\Omega) \subset \delta \mathcal{F}_{m,\chi}(\Omega)$. *Finally, we present a relationship between* $\delta \mathcal{F}_{m,\chi}(\Omega)$ *and the classes* $\delta \mathcal{W}_{m,\chi_k}(\Omega)$ *with* $k = 1, \ldots, m$.

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1. Introduction

Let $\beta = dd^c \|z\|^2$ be the canonical Kahler form of $\mathbb{C}^n$, where $d = \partial + \overline{\partial}$ and $d^c = \frac{\partial - \overline{\partial}}{4i}$, hence $dd^c = \frac{i}{2}\partial\overline{\partial}$. We denote by $dV_n = \frac{1}{n!}\beta^n$ the volume element of $\mathbb{C}^n$. The complex Monge-Ampère operator $(dd^c.)^n$ is well defined over the class of locally bounded plurisubharmonic (psh) functions, according to the fundamental work of Bedford and Taylor in [1-2]. In [13], Demailly generalized the work of Bedford and Taylor for the class of locally psh functions with bounded values near the boundary. In [7], Cegrell then introduced a general class $\mathcal{E}(\Omega)$ of psh functions on which the complex Monge-Ampère operator can be defined. Moreover, in [8] Cegrell introduced some subclasses of $\delta - PSH(\Omega)$ functions and gave some topology properties of these classes. Some elements of the theory of plurisubharmonic functions can be found in [1-5,7-9, 13]. On the other hand, recently, in [6-14] the authors have studied m-subharmonic functions which are extensions of the plurisubharmonic functions. The authors also studied the complex m-Hessian operator $H_m(.) = (dd^c.)^m \wedge \beta^{n-m}$ which is more general than the Monge-Ampère operator. In order to study the complex m-Hessian operator for m-subharmonic functions which are not locally bounded, in [11], Chinh introduced the Cegrell classes $\mathcal{F}_m(\Omega)$ and $\mathcal{E}_m(\Omega)$. Moreover, he proved that the complex m-Hessian operator is well defined in these classes. Some elements of the theory of m-subharmonic functions and the complex Hessian operator, that will be used throughout the note, can be found in [6-20],... Morever, by the same idea of S. Benelkourchi, V. Guedj and A. Zeriahi [4], in [15] by using m-capacity, the author investigated and introduced the classes $\mathcal{F}_{m,\chi}(\Omega)$ and $\mathcal{E}_{m,\chi}(\Omega)$ which are extention of the ones were introduced by Lu Hoang Chinh [10] for the class of m-subharmonic functions.

On the other hand, in [21], the authors introduced the vector space $\delta\mathcal{F}_m(\Omega)$ and equiped this space by a norm, which is defined by using the m-Hessian measure. They have proved $\delta\mathcal{F}_m(\Omega)$ is a Banach space and $\mathcal{F}_m(\Omega)$ is closed in this space. Moreover, they have shown that the topology defined by this norm is stronger than the convergence in m-capacity. Also, in [22], N. Thien has defined a quasi-norm on the vector space $\delta\mathcal{E}_p(\Omega)$ and proved that this vector space with this quasi-norm is a quasi-Banach space.

In this paper, we shall construct a new class $\mathcal{W}_{m,\chi}(\Omega)$ of weighted energy functions in the class of m-subharmonic functions and study some analysis properties of the delta m-subharmonic functons class. Then, by using the similar technique in [15], we introduce delta weighted energy class of m-subharmonic functions $\delta\mathcal{W}_{m,\chi}(\Omega)$ and prove this class is convex cone and stable under maximum and is contained in $\delta\mathcal{F}_{m,\chi}(\Omega)$ class. We also give a relationship between $\delta\mathcal{F}_{m,\chi}(\Omega)$ and the classes $\delta\mathcal{W}_{m,\chi_k}(\Omega)$ with $k = 1, \ldots, m$.

2. Preliminaries

Let Ω be a hyperconvex domain in $\mathbb{C}^n$. For a twice continuously differentiable real function $u \in C^2(\Omega)$, the second order differential at a fixed point $z_0 \in \Omega$

$$dd^c u = \frac{i}{2}\sum_{j,k}^{n} u_{j,\bar{k}} dz_j \wedge d\bar{z}_k$$

is a Hermitian quadratic form. After an appropriate unitary transformation of coordinates, it reduces to the diagonal form

$$dd^c u = \frac{i}{2}[\lambda_1 dz_1 \wedge d\bar{z}_1 + \cdots + \lambda_1 dz_n \wedge d\bar{z}_n],$$

where $\lambda_1(u), \ldots, \lambda_n(u)$ are the eigenvalues of the Hermitian matrix $(u_{j,\bar{k}})$, which are real, i.e.. $\lambda(u) = (\lambda_1(u), \ldots, \lambda_n(u)) \in \mathbb{R}^n$. It is easy to see that

$$(dd^c u)^m \wedge \beta^{n-m} = m!(n-m)! H_m(u)\beta^n, \forall 1 \le m \le n \qquad (2.1)$$

where $H_m(u) = \sum_{1 \le j_1 < \cdots < j_m \le n} \lambda_{j_1} \ldots \lambda_{j_m}$ is the Hessian of order m of the vector $\lambda(u) = (\lambda_1(u), \ldots, \lambda_n(u)) \in \mathbb{R}^n$. Thus, the operator $(dd^c u)^m \wedge \beta^{n-m}, \forall 1 \le m \le n,$ for twice continuously differentiable functions is related to the Hessian of the vector $\lambda = (\lambda_1, \ldots, \lambda_n)$.

2.1. The *m*-subharmonic functions

Now, we recall the class of *m*-subharmonic functions introduced and investigated in [9] recently. For $1 \le m \le n$, set

$$\hat{\Gamma}_m = \{\eta \in \mathbb{C}_{(1,1)} : \eta \wedge \beta^{n-1} \ge 0, \ldots, \eta^m \wedge \beta^{n-m} \ge 0\},$$

where $\mathbb{C}_{(1,1)}$ denotes the space of $(1,1)$-forms with constant coefficients.

Definition 2.1. Let u be a subharmonic function on an open subset $\Omega \subset \mathbb{C}^n$. Then, u is said to be a *m*-subharmonic function on Ω if for every $\eta_1, \ldots, \eta_{m-1}$ in $\hat{\Gamma}_m$ the inequality

$$dd^c u \wedge \eta_1 \wedge \ldots \wedge \eta_{m-1} \wedge \beta^{n-m} \ge 0,$$

holds in the sense of currents.

By $SH_m(\Omega)$ (resp. $SH_m^-(\Omega)$), we denote the cone of *m*-subharmonic functions (resp. negative *m*-subharmonic functions) on Ω. Now we recall the following definitions (see [9]).

For $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ and $1 \le m \le n$ define

$$S_m(\lambda) = \sum_{1 \le j_1 < \cdots < j_m \le n} \lambda_{j_1} \cdots \lambda_{j_m}.$$

Set

$$\Gamma_m = \{S_1 \ge 0\} \cap \{S_2 \ge 0\} \cap \cdots \cap \{S_m \ge 0\}.$$

By H, we denote the vector space of complex hermitian $n \times n$ matrices over $\mathbb{C}$. For $A \in \mathrm{H}$, let $\lambda(A) = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ be the eigenvalues of A. Set

$$S_m(A) = S_m(\lambda(A)).$$

As in [13], we define

$$\Gamma_m = \{A \in \mathrm{H} : \lambda(A) \in \Gamma_m\} = \{S_1 \ge 0\} \cap \cdots \cap \{S_m \ge 0\}.$$

Definition 2.2. Assume that $u_1, \ldots, u_p \in SH_m(\Omega) \cap L^\infty_{loc}(\Omega)$. Then the complex Hessian operator $H_m(u_1, \ldots, u_p)$ is defined inductively by

$$dd^c u_p \wedge \cdots \wedge dd^c u_1 \wedge \beta^{n-m} = dd^c(u_p dd^c u_{p-1} \wedge \cdots \wedge dd^c u_1 \wedge \beta^{n-m}).$$

From the definition of *m*-subharmonic functions and using arguments as in the proof of Theorem 2.1 in [2], we note that $H_m(u_1, \ldots, u_p)$ is a closed positive current of bidegree $(n-m+p, n-m+p)$ and this operator is continuous under decreasing sequences of locally bounded *m*-subharmonic functions. Hence, for $p = m$, $dd^c u_1 \wedge \cdots \wedge dd^c u_m \wedge \beta^{n-m}$ is a nonnegative Borel measure. In particular, when $u = u_1 = \cdots = u_m \in SH_m(\Omega) \cap L^\infty_{loc}(\Omega)$ the Borel measure

$$H_m(u) = (dd^c u)^m \wedge \beta^{n-m},$$

is well defined and is called the complex Hessian of u.

2.2. The $\mathcal{E}_m^0(\Omega)$ and $\mathcal{F}_m(\Omega)$ classes

Next, we recall the classes $\mathcal{E}_m^0(\Omega)$ and $\mathcal{F}_m(\Omega)$ introduced and investigated in [11]. First we give the following classes.

Let Ω be a bounded domain in $\mathbb{C}^n$. Then Ω is said to be *m*-hyperconvex if there exists a continuous *m*-subharmonic function $u : \Omega \to \mathbb{R}^-$ such that $\Omega_c = \{u < c\} \Subset \Omega$ for every $c < 0$. As above, every plurisubharmonic function is *m*-subharmonic with $m \ge 1$. So, every hyperconvex domain in $\mathbb{C}^n$ is *m*-hyperconvex. Let $\Omega \subset \mathbb{C}^n$ be a *m*-hyperconvex domain. Set

$$\mathcal{E}_m^0=\mathcal{E}_m^0(\Omega)=\{u\in SH_m^-(\Omega)\cap L^\infty(\Omega):\lim_{z\to\partial\Omega}u(z)=0,\ \int_\Omega H_m(u)<\infty\},$$

$$\mathcal{F}_m=\mathcal{F}_m(\Omega)=\{u\in SH_m^-(\Omega):\exists\ \mathcal{E}_m^0\ni u_j\searrow u,\ \sup_j\int_\Omega H_m(u_j)<\infty\},$$

$$\mathcal{E}_m=\mathcal{E}_m(\Omega)=\{u\in SH_m^-(\Omega):\forall z_0\in\Omega,\exists \text{ a neighborhood } \omega\ni z_0,\text{ and } \mathcal{E}_m^0\ni u_j\searrow u \text{ on } \omega,\ \sup_j\int_\Omega H_m(u_j)<\infty\},$$

where $H_m(u)=(dd^cu)^m\wedge\beta^{n-m}$ denotes the Hessian measure of $u\in SH_m^-(\Omega)\cap L^\infty(\Omega)$. From Theorem 3.14 in [11], it follows that if $u\in\mathcal{E}_m(\Omega)$, the complex Hessian $H_m(u)=(dd^cu)^m\wedge\beta^{n-m}$ is well defined and is a Radon measure on Ω. On the other hand, by Remark 3.6 in [11], we may give the following description of the class $\mathcal{E}_m(\Omega)$:

$$\mathcal{E}_m=\mathcal{E}_m(\Omega)=\{u\in SH_m^-(\Omega):\forall\ U\Subset\Omega,\exists\ v\in\mathcal{F}_m(\Omega),\ v=u \text{ on } U\}.$$

Now we recall the two weighted pluricomplex energy classes of plurisubharmonic functions defined in [2].

Definition 2.3. Let $\chi:\mathbb{R}^-\to\mathbb{R}^+$ be an decreasing function and $1\le m\le n$ and Ω is a bounded m-hyperconvex domain in $\mathbb{C}^n$. We define

$$\mathcal{F}_{m,\chi}(\Omega)=\{u\in SH_m^-(\Omega):\exists\ \{u_j\}\subset\mathcal{E}_m^0,\ u_j\searrow u\ on\ \Omega$$
$$\sup_j e_\chi^m(u_j)=\sup_j\int_\Omega\chi(u_j)(dd^cu_j)^m e^{n-m}<+\infty\}$$

and

$$\mathcal{E}_{m,\chi}(\Omega)=\{u\in SH_m^-(\Omega):\forall\ K\Subset\Omega,\exists v\in\mathcal{F}_{m,\chi},v=u\quad on\ K\}.$$

In the case $\chi(t)\equiv1$ for all $t<0$, we get $\mathcal{F}_{m,\chi}(\Omega)$ (resp. $\mathcal{E}_{m,\chi}(\Omega)$) coincide with the class $\mathcal{F}(\Omega)$ (resp. $\mathcal{E}(\Omega)$) in [12]. Moreover, by Proposition 4 in [18], $\mathcal{E}_{m,\chi}(\Omega)\subset\mathcal{E}_m(\Omega)$ provided that $\chi(2t)\le a\chi(t)$ for all $t<0$ with some $a>1$. So by using similar technique in [15] we can easily get the following result.

Proposition 2.4. If $\chi(2t)\le a\chi(t)$ for all $t<0$ with some $a>1$ then $\delta\mathcal{F}_{m,\chi}(\Omega)\subset\delta\mathcal{F}_m(\Omega)$ (resp. $\delta\mathcal{E}_{m,\chi}(\Omega)\subset\delta\mathcal{E}_m(\Omega)$).

3. The weighted capacity energy of *m*-subharmonic functions

In this section, we shall study some property of weighted capacity energy of m-subharmonic functions. As a consequence, we introduce some property of $\mathcal{E}_{m,\chi}(\Omega)$ and $\mathcal{F}_{m,\chi}(\Omega)$ class. Before getting the first result of this section, unless otherwise stated, throughout this section, to simplify the notation we will write " $A\lesssim B$" if there exists a constant $C>0$ such that $A\le CB$, and " $A\sim B$ " if there exists a constant $\delta_1,\delta_2>0$ such that $\delta_1A\le B\le\delta_2A$. The following class of functions were introduced in [18]

$$\mathcal{K}=\{\chi:\mathbb{R}^-\to\mathbb{R}^+ \text{ is a decreasing function such that } -t^2\chi''(t)\lesssim t\chi'(t)\lesssim\chi(t),\forall t<0\}.$$

As in [3], we put

$$e_{m,\chi}(u)=\int_\Omega\chi(u)(dd^cu)^m\wedge\beta^{n-m}.$$

Now we will show a basic properties of the weighted energy class of delta m-subharmonic functions that we will define in the following.

Definition 3.1. Assume that $\chi:\mathbb{R}^-\to\mathbb{R}^+$ is a function satisfies the condition $\chi(2t)\le a\chi(t),a>0$ and $\chi'(t)\le0,\forall t<0$. Then we define

$$\mathcal{W}_{m,\chi}=\mathcal{W}_{m,\chi}(\Omega):=\{\varphi\in SH_m(\Omega):e_\chi^m(\varphi)\sim\int_0^\infty-\chi'(-s)s^mC_m(\varphi<-s)ds<+\infty\}.$$

We put $\delta \mathrm{W}_{m,\chi} = \mathrm{W}_{m,\chi} - \mathrm{W}_{m,\chi}$.

Note that, the class $\mathrm{W}_{m,\chi}$ is extension of Cegrell class F_m in [10] and in unweighted space F_m [21].

Next, by Theorem 3.1 in [3], we shall prove the main result of this paper.

Theorem 3.2. The classes $\delta \mathrm{W}_{m,\chi}(\Omega)$ are convex and stable under maximum, i.e if $\varphi \in \delta \mathrm{W}_{m,\chi}(\Omega)$ and $\psi \in SH_m^-(\Omega)$ then $\max(\varphi,\psi) \in \delta \mathrm{W}_{m,\chi}(\Omega)$. Moreover, $\delta \mathrm{W}_{m,\chi}(\Omega) \subset \delta \mathrm{F}_{m,\chi}(\Omega)$.

Proof. We only need to prove the Proposition for $\mathrm{W}_{m,\chi}(\Omega)$.

First, we only prove the stability under maximum.

Step 1. We assume that $\chi(0)=0$. Set

$$\chi_j(t) := \chi(t) + \frac{(1-e^t)}{j}, \ t<0.$$

Then χ_j is a strictly decreasing function, $\chi < \chi_j < \chi + \frac{1}{j}$ and $\chi_j(2t) \le \max(a,2).\chi_j(t)$ for every $t<0$. Let $\omega = \max(\varphi,\psi)$, hence $\varphi \le \omega$. Moreover, since $\{\omega < -s\} \subset \{\varphi < -s\}$ for every $s>0$ so we have

$$\begin{aligned}
\int_\Omega \chi_j(\omega)(dd^c\omega)^m \wedge \beta^{n-m} &= -\int_0^{+\infty} \chi_{j'}(-s)(dd^c\omega)^m \wedge \beta^{n-m}\big(\{\omega<-s\}\big)ds \\
&\le -A\int_0^{+\infty} s^m \chi_{j'}(-s) C_m\big(\{\omega<-s\}\big)ds \\
&\le -2^m A\int_0^{+\infty} \chi_{j'}(-s)(dd^c\varphi)^m \wedge \beta^{n-m}\big(\{\varphi<-s/2\}\big)ds \\
&\le -A\int_0^{+\infty} s^m \chi_{j'}(-s) C_m\big(\{\varphi<-s\}\big)ds \\
&= A\int_\Omega \chi_j(2\varphi)(dd^c(2\varphi))^m \wedge \beta^{n-m} \\
&\le 2^m \max(a,2) A\int_\Omega \chi_j(\varphi)(dd^c\varphi)^m \wedge \beta^{n-m} \\
&\le 2^m \max(a,2) A\Big(\int_\Omega [\chi(\varphi)+\frac{1}{j}](dd^c\varphi)^m \wedge \beta^{n-m}\Big),
\end{aligned}$$

with A is a suitable constant.

Letting $j\to\infty$, we get

$$\begin{aligned}
\int_\Omega \chi(\omega)(dd^c\omega)^m \wedge \beta^{n-m} &\le 2^m \max(a,2) A\int_\Omega \chi(\varphi)(dd^c\varphi)^m \wedge \beta^{n-m} < +\infty \\
&= 2^m \max(a,2) A\int_0^\infty -\chi'(-s) s^m C_m(\varphi<-s)ds \\
&\sim e_\chi^m(\varphi) < +\infty.
\end{aligned}$$

So $\omega = \max(\varphi,\psi) \in \mathrm{W}_{m,\chi}(\Omega)$.

Step 2. In general case we set $\Phi_j(t) = \min(\chi(t); -jt)$ and approximately if necessary. Then Φ_j are decreasing functions such that $\Phi_j(0)=0$ and $\Phi_j \nearrow \chi$ on $(-\infty,0)$. By first case, we have

$$\int_\Omega \Phi_j(\omega)(dd^c\omega)^m \wedge \beta^{n-m} \le 2^m \max(a,2) A\int_\Omega \Phi_j(\varphi)(dd^c\varphi)^m \wedge \beta^{n-m}.$$

Letting $j \to \infty$, we obtain

$$\int_\Omega \chi(\omega)(dd^c\omega)^m \wedge \beta^{n-m} \le 2^m \max(a,2)A\int_\Omega \chi(\varphi)(dd^c\varphi)^m \wedge \beta^{n-m} < +\infty$$

$$= 2^m \max(a,2)A\int_0^\infty -\chi'(-s)s^m C_m(\varphi < -s)ds$$

$$\sim e_\chi^m(\varphi) < +\infty.$$

So, once again $\omega = \max(\varphi,\psi) \in \mathcal{W}_{m,\chi}(\Omega)$.

The convexity of $\mathcal{W}_{m,\chi}(\Omega)$ follows from the following: if $\varphi,\psi \in \mathcal{W}_{m,\chi}(\Omega)$ and $0 \le t_0 \le 1$ then

$$\{t_0\varphi + (1-t_0)\psi < -s\} \subset \{\varphi < -s\} \cup \{\psi < -s\}.$$

As above, we can assume that $\chi(0) = 0$, so we have

$$\int_\Omega \chi(t_0\varphi + (1-t_0)\psi)(dd^c(t_0\varphi + (1-t_0)\psi))^m \wedge \beta^{n-m}$$

$$\le \int_\Omega \chi_j(t_0\varphi + (1-t_0)\psi)(dd^c(t_0\varphi + (1-t_0)\psi))^m \wedge \beta^{n-m}$$

$$\le -A\int_0^{+\infty} s^m \chi_{j'}(-s)C_m\big(\{\varphi < -s\}\big)ds - A\int_0^{+\infty} s^m \chi_{j'}(-s)C_m\big(\{\psi < -s\}\big)ds$$

$$\le 2^m \max(a,2)A[\int_\Omega (\chi(\varphi) + \frac{1}{j})(dd^c\varphi)^m \wedge \beta^{n-m} + \int_\Omega (\chi(\psi) + \frac{1}{j})(dd^c\psi)^m \wedge \beta^{n-m}].$$

Letting $j \to \infty$ this yields

$$\int_\Omega \chi(t_0\varphi + (1-t_0)\psi)(dd^c(t_0\varphi + (1-t_0)\psi))^m \wedge \beta^{n-m}$$

$$\le 2^m \max(a,2)A[\int_\Omega \chi(\varphi)(dd^c\varphi)^m \wedge \beta^{n-m} + \int_\Omega \chi(\psi)(dd^c\psi)^m \wedge \beta^{n-m}]$$

$$= 2^m \max(a,2)A[\int_0^\infty -\chi'(-s)s^m C_m(\varphi < -s)ds + \int_0^\infty -\chi'(-s)s^m C_m(\psi < -s)ds]$$

$$\sim e_\chi^m(\varphi) + e_\chi^m(\psi) < +\infty.$$

Finally, assume $\varphi \in \mathcal{W}_{m,\chi}(\Omega)$. We can assume without loss of generality $\varphi \le 0$ and $\chi(0) = 0$.

Set $\varphi_j := \max(\varphi, -j)$. It follows from Lemma 1 in [2] that

$$\int_\Omega \chi(\varphi_j)(dd^c\varphi_j)^m \wedge \beta^{n-m} = \int_0^{+\infty} -\chi'(-s)(dd^c\varphi_j)^m \wedge \beta^{n-m}(\varphi_j < -s)ds$$

$$\le A\int_0^{+\infty} -\chi'(-s)s^m C_m(\varphi < -s)ds < +\infty.$$

This shows that $\varphi \in \mathcal{F}_{m,\chi}(\Omega)$. □

Corollary 3.3. Assume that the assumption of Proposition 3.2 is satisfied. Then the classes $\delta\mathcal{W}_{m,\chi}(\Omega)$ is a convex cone in $\delta\mathcal{F}_{m,\chi}(\Omega)$.

Before coming to the following result, we set $\chi_0(t) = \chi(t)$ and for each $k \ge 1$, let $\chi_k(t) = -\int_0^t \chi_{k-1}(x)dx$. If $\chi \in \mathcal{K}$ then it is easy to check that $\chi_k \in \mathcal{K}$ and $\chi(t)(-t)^k \lesssim \chi_k(t) \lesssim \chi(t)(-t)^k$.

Finally, we give a new relationship between $\delta\mathcal{F}_{m,\chi}(\Omega)$ and $\delta\mathcal{W}_{m,\chi_k}(\Omega')$ class.

Theorem 3.4. Let Ω be a m-hyperconvex domain in $\mathbb{C}^n$ and $1 \le m \le n$. Assume that $\chi \in \mathcal{K}$ such that $\chi''(t) \ge 0, \forall t < 0$. Then for $\Omega' \Subset \Omega$, we have

$$\delta\mathcal{F}_{m,\chi}(\Omega) \subset \bigcap_{k=1}^m \delta\mathcal{W}_{m,\chi_k}(\Omega').$$

Proof. In order to prove the Proposition, it is necessary and sufficient that there exists a constant $C = C(\Omega')$ such that

$$\int_{\Omega'} \chi(u)|u|^k (dd^c u)^{m-k} \wedge \beta^{n-m+k} \leq C\int_{\Omega} \chi(u)(dd^c u)^m \wedge \beta^{n-m},$$

holds for $F_{m,\chi}(\Omega)$. Indeed, we shall begin with showing that the (3.1) holds for $u \in E_m^0(\Omega)$

To do that, we choose $R > 0$ large enough such that $\|z\|^2 \leq R^2$ on Ω. We fixed $\varphi \in E_m^0(\Omega)$ and $A > 0$ such that $\|z\|^2 - R^2 \geq A\varphi$ on Ω'. Set $h = \max(\|z\|^2 - R^2; A\varphi)$ then $h \in E_m^0(\Omega)$ and $dd^c h = dd^c\|z\|^2 = \beta$ on Ω'. Then, we have the following estimates

$$\int_{\Omega'} \chi(u)|u|^k (dd^c u)^{m-k} \wedge (dd^c h)^k \wedge \beta^{n-m}$$

$$\leq \int_{\Omega} \chi(u)|u|^k (dd^c u)^{m-k} \wedge (dd^c h)^k \wedge \beta^{n-m}$$

ˆ $$\int_{\Omega} \chi_k(u)(dd^c u)^{m-k} \wedge (dd^c h)^k \wedge \beta^{n-m}.$$

By integration by parts we have

$$\int_{\Omega} \chi_k(u)(dd^c u)^{m-k} \wedge (dd^c h)^k \wedge \beta^{n-m}$$

$$= \int_{\Omega} h(dd^c u)^{m-k} dd^c \chi_k(u) \wedge (dd^c h)^{k-1} \wedge \beta^{n-m}$$

$$= \int_{\Omega} h(dd^c u)^{m-k}[\chi_k''(u) du \wedge d^c u + \chi_k'(u) dd^c u] \wedge (dd^c h)^{k-1} \wedge \beta^{n-m}$$

$$\leq \int_{\Omega} h\chi_k'(u)(dd^c u)^{m-k+1} \wedge (dd^c h)^{k-1} \wedge \beta^{n-m}$$

$$\leq \|h\|_{L^\infty(\Omega)} \int_{\Omega} \chi_{k-1}(dd^c u)^{m-k+1} \wedge (dd^c h)^{k-1} \wedge \beta^{n-m}$$

$$\leq \cdots\cdots\cdots$$

$$\leq \|h\|^k_{L^\infty(\Omega)} \int_{\Omega} \chi(u)(dd^c u)^m \wedge \beta^{n-m}.$$

Hence, if we set $C = C(\Omega') = k!\|h\|^k_{L^\infty(\Omega)}$ then

$$C\int_{\Omega} \chi(u)(dd^c u)^m \wedge \beta^{n-m} \geq \int_{\Omega} \chi(u)|u|^k (dd^c u)^{m-k} \wedge (dd^c h)^k \wedge \beta^{n-m}$$

$$\geq \int_{\Omega'} \chi(u)|u|^k (dd^c u)^{m-k} \wedge (dd^c h)^k \wedge \beta^{n-m}$$

$$= \int_{\Omega'} \chi(u)|u|^k (dd^c u)^{m-k} \wedge (dd^c \|z\|^2)^k \wedge \beta^{n-m}.$$

Next, we prove (3.1) holds for $u \in F_{m,\chi}(\Omega)$. Indeed, we take $u_j \in E_m^0(\Omega), u_j \searrow u$ on Ω such that

$$\sup_{j \geq 1} \int_{\Omega} \chi(u_j)(dd^c u_j)^m \wedge \beta^{n-m} < +\infty.$$

By dominated convergence theorem and $(dd^c u_j)^{m-k} \wedge (dd^c\|z\|^2)^{n-m+k}$ is weakly convergent to $(dd^c u)^{m-k} \wedge (dd^c\|z\|^2)^{n-m+k}$ in the sense of currents, we have

$$
\begin{aligned}
&\int_{\Omega'} \chi(u)\,|u|^k\,(dd^c u)^{m-k} \wedge (dd^c \|z\|^2)^{n-m+k} \\
&\quad \le \liminf_j \int_{\Omega'} \chi(u_j)\,|u_j|^k\,(dd^c u_j)^{m-k} \wedge (dd^c \|z\|^2)^{n-m+k} \\
&\quad \le \liminf_j \int_{\Omega} \chi(u_j)\,|u_j|^k\,(dd^c u_j)^{m-k} \wedge (dd^c h)^k \wedge (dd^c \|z\|^2)^{n-m} \\
&\quad \le C \sup_j \int_{\Omega} \chi(u_j)(dd^c u_j)^m \wedge (dd^c \|z\|^2)^{n-m} < +\infty.
\end{aligned}
$$

Finally, the assertion of the Proposition follows from (3.1) and this completes the proof. □

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MỘT SỐ TÍNH CHẤT CỦA MỘT SỐ LỚP HÀM DELTA *m*-ĐIỀU HÒA DƯỚI

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Tóm tắt: *Mục đích của bài báo này là giới thiệu một lớp hàm delta m-điều hòa dưới với năng lượng có trọng* $\delta \mathrm{W}_{m,\chi}(\Omega)$. *Đồng thời chúng tôi cũng chỉ ra lớp hàm đã đưa ra là một nón lồi, ổn định với phép toán lấy max và* $\delta \mathrm{W}_{m,\chi}(\Omega) \subset \delta \mathrm{F}_{m,\chi}(\Omega)$. *Cuối cùng chúng tôi đưa ra một mối quan hệ giữa* $\delta \mathrm{F}_{m,\chi}(\Omega)$ *và các lớp hàm* $\delta \mathrm{W}_{m,\chi_k}(\Omega)$ *với* $k = 1, \ldots, m$..

Từ khóa: *Hàm đa điều hòa dưới, hàm m-điều hòa dưới, delta m-điều hòa dưới, miền m-siêu lồi, toán tử m-Hessian, toán tử Monge-Ampere, hàm trọng, m-dung tích.*

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